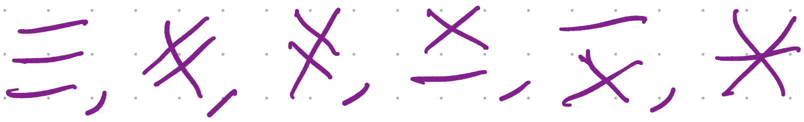
GTLA Lecture 10.09.2020

A choice of generators and relations for S_3

$S_3 = \{$  $\}$

$= \{ 1, r, r^2, sr, sr^2, s \}$

Let

$1 =$  $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

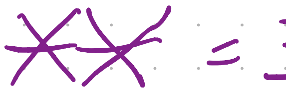
$s =$  $= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$r =$  $= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

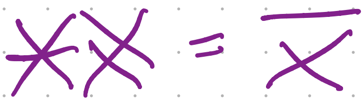
So

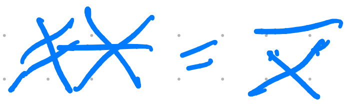
$r^2 =$ 

$r^3 = r^2 \cdot r =$  $= 1.$

$s^2 =$  $= 1.$

$sr =$ 

$sr^2 =$ 

$rs =$  $= sr^2$

Note: $r^2 = r^{-1}$ since $r^3 = 1$.

$r \cdot r^2 = r^3 = 1$ So $r^2 = r^{-1}$.

$S_3 = \{1, r, r^2, s, sr, sr^2\}$
is generated by r and s
with relations

$$s^2 = 1, \quad r^3 = 1, \quad rs = sr^2 = sr^{-1}$$

Let G be a group, S a set.
An action of G on S is a function

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$ such that

(a) If $g_1, g_2 \in G$ and $x \in S$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x.$$

(b) If $x \in S$ then $1 \cdot x = x$

S (with its action of G) is a G-set (analogous to an $\mathbb{F}$-vector space)

Let S be a G-set. Let $x \in S$.
The stabiliser of x is

$\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$.

The orbit of x is

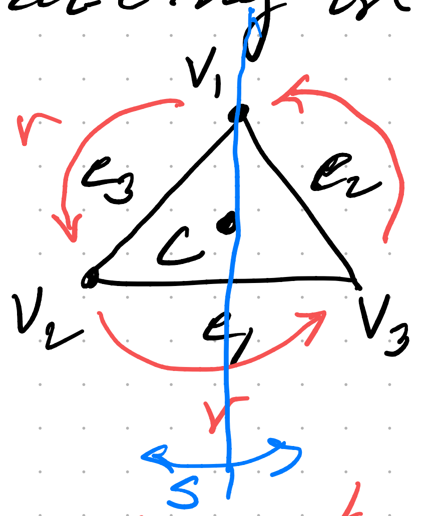
$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

Analogy:
Let $f : G \to H$ be a homomorphism.

$$\ker f = \{g \in G \mid f(g) = 1\}$$

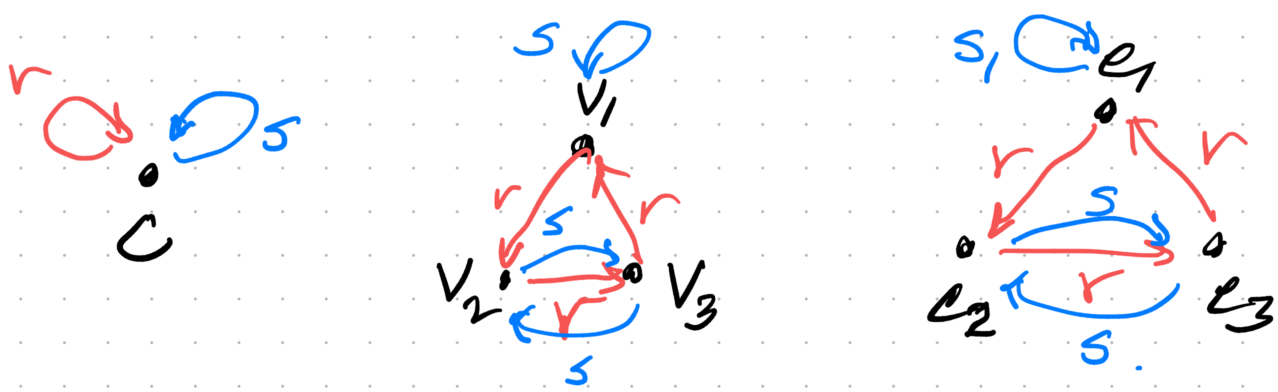
$$\mathrm{im} f = \{f(g) \mid g \in G\}.$$

Example $G = S_3 = \{1, r, r^2, s, sr, sr^2\}$
acting on a triangle.



$S = \{c, v_1, v_2, v_3, e_1, e_2, e_3\}$

r is rotation by $\frac{2\pi}{3}$.
s is reflection in vertical axis.



Orbits $G = S_3$

$S = \{c, v_1, v_2, v_3, e_1, e_2, e_3\}$

$G\cdot c = \{c\}$

$G\cdot v_1 = \{v_1, v_2, v_3\}$
$= G\cdot v_2 = \{v_2, v_1, v_3\}$
$= G\cdot v_3 = \{v_3, v_1, v_2\}$

$G\cdot e_1 = \{e_1, e_2, e_3\}$
$= G\cdot e_2$
$= G\cdot e_3$.

Stabilisers

$\mathrm{Stab}_G(c) = \{1, r, r^2, s, sr, sr^2\}$

$\mathrm{Stab}_G(v_1) = \{1, s\}$

$\mathrm{Stab}_G(v_2) = \{1, sr\}$

$\mathrm{Stab}_G(v_3) = \{1, sr^2\}$

$\mathrm{Stab}_G(e_1) = \{1, s\}$

$\mathrm{Stab}_G(e_2) = \{1, sr\}$

$\mathrm{Stab}_G(e_3) = \{1, sr^2\}$

rs
$= sr^2$

Proposition Let G be a group and let S be a G-set. Let $x \in S$ and $g \in G$.

(a) $G\cdot x$ is a subset of S

(b) $\mathrm{Stab}_G(x)$ is a subgroup of G.

(c) $\mathrm{Stab}_G(g\cdot x) = g\,\mathrm{Stab}_G(x)\,g^{-1}$

$\mathrm{Stab}_G(g^{-1}x) = g^{-1}\mathrm{Stab}_G(x)g$.

(d) The orbits partition S.

<u>Proof</u> (d)

To show: (da) $\bigcup_{x\in S} G\cdot x = S$.

(db) If $x, y\in S$ and $G\cdot x \cap G\cdot y \neq \emptyset$ then $G\cdot x = G\cdot y$.

(da) To show: (daa) $\bigcup_{x\in S} G\cdot x \subseteq S$

(dab) $S\subseteq \bigcup_{x\in S} G\cdot x$.

(daa) Since $G\cdot x = \{g\cdot x \mid g\in G\}\subseteq S$ then $\bigcup_{x\in S} G\cdot x \subseteq S$.

(dab) To show: If $z\in S$ then there exists $x\in S$ such that $z\in G\cdot x$.

Assume $z\in S$.

To show: There exists $x\in S$ such that $z\in G\cdot x$.

Let $x = z$

Since $z = 1 \cdot z = 1 \cdot x \in G \cdot x$

then $z \in G \cdot x$.

So $\bigcup_{x \in S} G \cdot x = S$.

(db) To show: If $x, y \in S$ and $G \cdot x \cap G \cdot y \neq \emptyset$ then $G \cdot x = G \cdot y$.

Assume $x, y \in S$ and $G \cdot x \cap G \cdot y \neq \emptyset$.

Then there exists $z \in S$ such that $z \in G \cdot x$ and $z \in G \cdot y$.

So there exist $g_1, g_2 \in G$ such that $z = g_1 \cdot x$ and $z = g_2 \cdot y$.

So $x = g_1^{-1} \cdot z = g_1^{-1} g_2 \cdot y$

and $y = g_2^{-1} \cdot z = g_2^{-1} g_1 \cdot x$.

To show: $G \cdot x = G \cdot y$.

To show: (dba) $G \cdot x \subseteq G \cdot y$

(dbb) $G \cdot y \subseteq G \cdot x$.

(dba) To show: If $a \in G \cdot x$ then $a \in G \cdot y$.

Assume $a \in G \cdot x$.

To show: $a \in G \cdot y$.

Since $a \in G \cdot x$ then there exists $g \in G$ such that $a = g \cdot x$.

So $a = g \cdot x = g\, g_1^{-1} g_2 \cdot y \in G \cdot y$.

So $G \cdot x \subseteq G \cdot y$.

(dbb) To show: If $b \in G \cdot y$ then $b \in G \cdot x$.

Let $b \in G \cdot y$. Then there exists $k \in G$ such that $b = k \cdot y$.

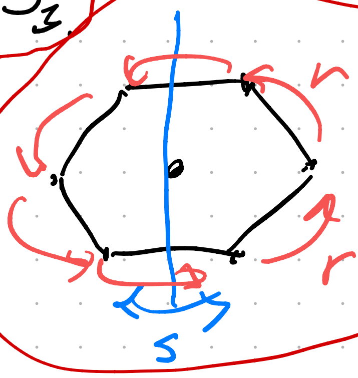
So $b = k \cdot y = k \cdot g_2^{-1} g_1 \cdot x = (k g_2^{-1} g_1) \cdot x \in G \cdot x$

So $G \cdot y \subseteq G \cdot x$.

So $G \cdot x = G \cdot y$.

So the orbits partition S. $\square$

 $D_3 = S_3$.



D_6.

$S_6 \supseteq D_6$.

$r =$ 

$s =$