Eigenvectors and A-invariant subspaces

Let $\mathbb{F}$ be a field and $A \in M_n(\mathbb{F})$. Let $p \in \mathbb{F}^n$ with $p \neq 0$. Show that p is an eigenvector of A if and only if $\mathbb{F}p$ is A-invariant.

Proof $\Rightarrow$ Assume p is an eigenvector.

To show: $\mathbb{F}p$ is A-invariant.

Since p is an eigenvector then there exists $\lambda \in \mathbb{F}$ such that $Ap = \lambda p$.

To show: If $u \in \mathbb{F}p$ then $Au \in \mathbb{F}p$.

Assume $u \in \mathbb{F}p$.

Then there exists $c \in \mathbb{F}$ such that $u = cp$.

To show: $Au \in \mathbb{F}p$.

$Au = A(cp) = cAp$
$= c\lambda p = (c\lambda)p \in \mathbb{F}p.$

So $\mathbb{F}p$ is A-invariant.

$\Leftarrow$ Assume $\mathbb{F}p$ is A-invariant

To show: p is an eigenvector of A.

Since $\mathbb{F}p$ is A-invariant and $p \in \mathbb{F}p$ then $Ap \in \mathbb{F}p$

So there exists $\lambda \in \mathbb{F}$ such that $Ap = \lambda p$.

So p is an eigenvector of A //.

Eigenvectors and nullspaces

Let $\mathbb{F}$ be a field and let $A \in M_n(\mathbb{F})$. Let $\lambda \in \mathbb{F}$ and $p \in \mathbb{F}^n$ with $p \neq 0$. Show that
p is an eigenvector of eigenvalue λ
if and only if $p \in \ker(\lambda - A)$.

Proof $\Rightarrow$ Assume p is an eigenvector of A of eigenvalue λ. Then $Ap = \lambda p$.

So $\lambda p - Ap = 0$. So $(\lambda - A)p = 0$.

So $p \in \ker(\lambda - A)$.

$\Leftarrow$ Assume $p \in \ker(\lambda - A)$.

Then $(\lambda - A)p = 0$.

So $\lambda p - Ap = 0$ and $Ap = \lambda p$.

So p is an eigenvector of eigenvalue λ. //

Distinct eigenvalues give linearly independent eigenvectors

Let $\mathbb{F}$ be a field and let $A \in M_n(\mathbb{F})$. Assume $p_1, \dots, p_k \in \mathbb{F}^n$ are eigenvectors of A with eigenvalue $\lambda_1, \dots, \lambda_k$. Show that if $\lambda_1, \dots, \lambda_k$ are all distinct then $p_1, \dots, p_k$ are linearly independent.

Proof Assume $p_1, \dots, p_k$ are eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_k$ and $\lambda_1, \dots, \lambda_k$ are all distinct.

To show: $p_1, \dots, p_k$ are linearly independent.

To show: If $c_1, \dots, c_k \in \mathbb{F}$ and $c_1 p_1 + \cdots + c_k p_k = 0$ and if $j \in \{1, \ldots, k\}$ then $c_j = 0$.

Assume $c_1, \ldots, c_k \in \mathbb{F}$ and

$c_1 p_1 + \cdots + c_k p_k = 0$ and $j \in \{1, \ldots, k\}$.

To show: $c_j = 0$.

We know: $0 = c_1 p_1 + \cdots + c_k p_k$

If M is a matrix, $M \in M_n(\mathbb{F})$

$$0 = M(c_1 p_1 + \cdots + c_k p_k)$$

Let

$$M = (\lambda_1 - A)(\lambda_2 - A) \cdots (\lambda_{j-1} - A) \cdot (\lambda_{j+1} - A) \cdots (\lambda_k - A).$$

Note that $(\lambda_r - A)(\lambda_s - A) = (\lambda_s - A)(\lambda_r - A)$.

So $M p_1 = (\lambda_k - A) \cdots (\lambda_1 - A) p_1$

$= (\lambda_k - A) \cdots (\lambda_1 - \lambda_1) p_1 = 0$

(since $A p_1 = \lambda_1 p_1$ and $\lambda_1 - \lambda_1 = 0$).

$M p_2 = (\lambda_k - A) \cdots (\lambda_1 - A)(\lambda_2 - A) p_2$

$= (\lambda_k - A) \cdots (\lambda_1 - A)(\lambda_2 - \lambda_2) p_2 = 0.$

So $Mp_1 = 0$, $Mp_2 = 0$, $\ldots$, $M_{j-1}p_{j-1}$,

$M_{j+1}p_{j+1} = 0$, $\ldots$, $Mp_k = 0$.

$$0 = M(c_1p_1 + \cdots + c_kp_k)$$

$$= c_1Mp_1 + \cdots + c_{j-1}Mp_{j-1}$$

$$+ c_jMp_j$$

$$+ c_{j+1}Mp_{j+1} + \cdots + c_kMp_k.$$

$$= 0 + \cdots + 0$$

$$+ c_jMp_j$$

$$+ 0 + \cdots + 0$$

Since

$$Mp_j = (\lambda_1 - A)\cdots(\lambda_{j-1} - A)(\lambda_{j+1} - A)\cdots(\lambda_k - A)$$

$$\cdot p_j$$

and $Ap_j = \lambda_j p_j$ then

$$Mp_j = (\lambda_1 - \lambda_j)(\lambda_2 - \lambda_j)\cdots(\lambda_{j-1} - \lambda_j)$$

$$\cdot(\lambda_{j+1} - \lambda_j)\cdots(\lambda_k - \lambda_j)p_j$$

Then ~~Since~~ $\lambda_1 - \lambda_j \neq 0$, $\lambda_2 - \lambda_j \neq 0$, $\ldots$

$\cdots (\lambda_k - \lambda_j) \neq 0$

because $\lambda_1, \ldots, \lambda_k$ are distinct.

$$0 = c_j M p_j$$
$$= c_j (\lambda_1 - \lambda_j) \cdots (\lambda_{j-1} - \lambda_j)$$
$$\cdot (\lambda_{j+1} - \lambda_j) \cdots (\lambda_k - \lambda_j) p_j.$$

Since $\lambda_1 - \lambda_j \neq 0, \ldots, \lambda_k - \lambda_j \neq 0$ then $c_j = 0$.

So $p_1, \ldots, p_k$ are linearly independent. //

If $\mathbb{F}$ is algebraically closed then $A \in M_n(\mathbb{F})$ has an eigenvector

Let $\mathbb{F}$ be a field and assume $\mathbb{F}$ is algebraically closed. Let $A \in M_n(\mathbb{F})$. Show that A has an eigenvector (over $\mathbb{F}$).

Proof Assume $\mathbb{F}$ is algebraically closed and $A \in M_n(\mathbb{F})$.

Since $\mathbb{F}$ is algebraically closed then the characteristic polynomial $\det(x-A)$ has a root.

So there exists $\lambda \in \mathbb{F}$ such that $\det(\lambda - A) = 0$.

Since $\det(\lambda - A) = 0$ then

$\ker(\lambda - A) \neq 0$.

So there exist $p \in \ker(\lambda - A)$ with $p \neq 0$.

Then p is an eigenvector of A ($(\lambda - A)p = 0$ gives $Ap = \lambda p$). //.