GTLA Lecture 24.09.2020

Inner products.

The <u>standard inner product on $\mathbb{R}^n$</u> is

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(\vec{u}, \vec{v}) \longmapsto \vec{u} \cdot \vec{v} \text{ given by}$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 \overline{v_1} + u_2 \overline{v_2} + \cdots + u_n \overline{v_n}.$$

The <u>standard inner product on $\mathbb{C}^n$</u>

$$\mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$
$$(\vec{u}, \vec{v}) \longmapsto \vec{u} \cdot \vec{v}. \text{ given by}$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 \overline{v_1} + u_2 \overline{v_2} + \cdots + u_n \overline{v_n}$$

where $\overline{x+iy} = x - iy$ for

$x + iy \in \mathbb{C}$

<u>Note</u>: If $x \in \mathbb{R}$ then $\overline{x} = x$

Let $\mathbb{F}$ be a field. Let $\bar{\ }: \mathbb{F} \to \mathbb{F}$ satisfy: if $c, c_1, c_2 \in \mathbb{F}$ then

$$\overline{c_1 + c_2} = \bar{c}_1 + \bar{c}_2, \quad \overline{c_1 \cdot c_2} = \bar{c}_2 \cdot \bar{c}_1 \text{ and}$$
$$\bar{\bar{c}} = c. \text{ and } \bar{1} = 1.$$

Favorite example:

$\mathbb{F} = \mathbb{C}$ and $\bar{\ }: \mathbb{C} \to \mathbb{C}$ is complex conjugation.

Even more favourite example:

$\mathbb{F}$ is a field $\quad \bar{\ }: \mathbb{F} \to \mathbb{F}$ so that
$\qquad c \mapsto c$

$\bar{c} = c$ ($\bar{\ }$ is the identity).

Let $\mathbb{F}$ be a field and $\bar{\ }: \mathbb{F} \to \mathbb{F}$ as above (an involutive automorphism). Let V be an $\mathbb{F}$-vector space.

A <u>sesquilinear form on V</u> is a function
$$V \times V \to \mathbb{F}$$
$$(u, v) \mapsto \langle u, v \rangle$$
such that

(1) If $v_1, v_2, u \in V$ then

$$\langle v_1 + v_2, u \rangle = \langle v_1, u \rangle + \langle v_2, u \rangle.$$

(2) If $v, u_1, u_2 \in V$ then

$$\langle v, u_1 + u_2 \rangle = \langle v, u_1 \rangle + \langle v, u_2 \rangle.$$

(3) If $c \in \mathbb{F}$ and $u, v \in V$ then

$$\langle cv, u \rangle = c \langle v, u \rangle$$

(4) If $c \in \mathbb{F}$ and $u, v \in V$ then

$$\langle v, cu \rangle = \bar{c} \langle v, u \rangle$$

A sesquilinear form is <u>Hermitian</u> if it satisfies:

If $u, v \in V$ then $\langle u, v \rangle = \overline{\langle v, u \rangle}$

(If $\bar{\ } = \mathrm{id}$ then Hermitian is called symmetric)

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form. Let $B = \{b_1, \dots, b_n\}$ be a basis of V.

The Gram matrix of $\langle , \rangle$ with respect to B is $G_B \in M_n(\mathbb{F})$ given by

$$G_B(i,j) = (\langle b_i, b_j \rangle).$$

Example Dot product on $\mathbb{R}^n$ with basis $E = \{e_1, \dots, e_n\}$ where

$$e_i = \begin{array}{c} \\ \\ \\ i\text{th} \\ \\ \\ \end{array}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{then } e_i \cdot e_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

So the Gram matrix of dot product with respect to E is

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form.

Let $W \subseteq V$ be a subspace of V.

The <u>orthogonal to W</u> is

$$W^\perp = \{ v \in V \mid \text{if } w \in W \text{ and } \langle v, w \rangle = 0 \}.$$

(this is "like a kernel of W").

The subspace W is <u>nonisotropic</u> if

$$W \cap W^\perp = 0.$$

Heading towards the following:

<u>Theorem</u> If W is finite dim'l and $W \cap W^\perp = 0$ then

$$V = W \oplus W^\perp.$$

(this is "like a block decomposition").

Before we do this theorem...

<u>Proposition</u> Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form. Then $\langle , \rangle$ satisfies

if $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$

(no isotropic vectors)

if and only if $\langle , \rangle$ satisfies

if $W \subseteq V$ is a subspace then

$$W \cap W^\perp = 0.$$

(no isotropic subspaces).

If $\langle , \rangle$ satisfies these conditions then $\langle , \rangle$ is

<u>nonisotropic</u>.

<u>Proof</u> $\Rightarrow$ Assume $\langle , \rangle$ satisfies the red condition.

To show: If $W \subseteq V$ is a subspace then $W \cap W^\perp = 0$.

Assume $W \subseteq V$ is a subspace.

To show: $W \cap W^\perp = 0$.

To show: If $w \in W \cap W^\perp$ then $w = 0$.

Assume $w \in W \cap W^\perp$. Then

$0 = \langle w, w \rangle$, So $w = 0$.

So $W \cap W^\perp = 0$.

$\Leftarrow$ Assume the orange condition holds.

To show: If $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$.

Assume $v \in V$ and $\langle v, v \rangle = 0$.

To show: $v = 0$.

Let $W = \mathbb{F}v = \text{span}\{v\}$. ('dim 1' subspace)

Since $\langle v, cv \rangle = \bar{c}\langle v, v \rangle = \bar{c} \cdot 0 = 0$ for any $c \in \mathbb{F}$ then

$v \in W^\perp$

So $v \in W^\perp \cap W$.

By the orange condition $W^\perp \cap W = 0$ and so $v = 0$. //

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form. Let $W \subseteq V$ a subspace of V.

Assume W is finite dimensional

Let $k = \dim(W)$ and $B = \{w_1, \ldots, w_k\}$.

The <u>dual basis</u> to B is

$\{w^1, w^2, \ldots, w^k\}$ such that

$$\langle w^i, w_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

<u>Proposition</u> The following are equivalent:

(1) $\{w^1, w^2, \ldots, w^k\}$ exists

(2) G_B is invertible, where

$$G_B(i,j) = \langle w_i, w_j \rangle.$$ (The Gram matrix)

(3) $W \cap W^\perp = 0$.

<u>Proof Sketch</u> (1) $\Leftrightarrow$ (2)

Define

$$w^i = \sum_{\ell=1}^{k} G_B^{-1}(i,\ell)\, w_\ell.$$

(this defines the dual basis elements $w^1, \ldots, w^k$ from G_B^{-1}, the inverse of G_B.)

(the expansion of w^i in terms of $w_1, \ldots, w_k$ determines the entries of G_B^{-1}.)

Usually an inner product is sesquilinear and nonisotropic and pos. definite (if $v \in V$ $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$).