GTLA Lecture 30.10.2020

Let $V = M_n(\mathbb{F})$

In our mind $n=2$ and V has basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \{ E_{11}, E_{12}, E_{21}, E_{22} \}$$

where E_{ij} is the matrix with 1 in (i,j) entry and 0 everywhere else.

For general n,

$$B = \{ E_{ij} \mid i,j \in \{1, \dots, n\} \}$$

is a basis of V ("matrix units").

We defined

$$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{F} \text{ by}$$

$$\langle X, Y \rangle = \operatorname{tr}(XY).$$

Proposition

(a) If $X \in V$ and $X \neq 0$

then there exists $y \in V$ such that
$\langle x, y \rangle \neq 0$ (nondegeneracy).

(b) $V^\perp = 0$ and $V \cap V^\perp = 0$.

<u>Proof of (b)</u> To show: $V^\perp = 0$.

To show: If $x \in V^\perp$ then $x = 0$.

To show: If $x \neq 0$ then $x \notin V^\perp$

Assume $x \neq 0$

To show: $x \notin V^\perp$.

By (a), we know there exists $y \in V$ such that $\langle x, y \rangle \neq 0$.

So $x \notin V^\perp$.

So $V^\perp = \{0\}$.

So $V \cap V^\perp = V \cap \{0\} = \{0\}$ //.

$$\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \right)$$

$$= \mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$ (a nonzero vector of length 0).

Gram-Schmidt does not usually work for this $\langle , \rangle$.

However,

$$\langle E_{ij}, E_{kl} \rangle = tr(E_{ij} E_{kl})$$

$$= \begin{cases} tr(E_{il}), & \text{if } j=k \\ tr(0), & \text{if } j \neq k \end{cases}$$

$$= \begin{cases} 1, & \text{if } j=k \text{ and } i=l \\ 0, & \text{otherwise.} \end{cases}$$

So $\langle E_{ij}, E_{ji} \rangle = 1$ and

otherwise we get 0.

So if

$B = \{ E_{ij} \mid i,j \in \{1, \dots, n\} \}$

then THE DUAL BASIS

is $C = \{ E_{ji} \mid j,i \in \{1, \dots, n\} \}$.

(C is B but ordered differently).

Let $A \in M_n(\mathbb{F})$.

$A = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$.

Define

$$f : V \longrightarrow V$$
$$X \longmapsto AX - XA.$$

Show that f is a linear transformation.

Proof

(a) To show: If $X, Y \in V$ then
$$f(X+Y) = f(X) + f(Y).$$

(b) To show: If $X \in V$ and $c \in \mathbb{F}$ then $f(cX) = cf(X)$.

(a) Assume $X, Y \in V$.

To show: $f(X+Y) = f(X) + f(Y)$.

$$\begin{aligned} f(X+Y) &= A(X+Y) - (X+Y)A \\ &= AX + AY - XA - YA \\ &= AX - XA + AY - YA \end{aligned}$$

$= f(X) + f(Y)$

(b) Assume $X \in V$ and $c \in \mathbb{F}$.

To show: $f(cX) = cf(X)$.

$$f(cX) = A(cX) - (cX)A$$
$$= cAX - cXA$$
$$= c(AX - XA) = cf(X) \; //.$$

<u>Example</u> $n = 2$. $\quad f: V \to V$

$$A = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}.$$

$V = M_2(\mathbb{F})$ and $f(X) = AX - XA$.

Use the basis for V given by

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
$$= \{ E_{11}, E_{12}, E_{21}, E_{22} \}.$$

The matrix of f with respect to B is

$$f_{BB} = \begin{pmatrix} 0 & 3 & 12 & 0 \\ -12 & 12 & 0 & 12 \\ -3 & 0 & -12 & 3 \\ 0 & -3 & -12 & 0 \end{pmatrix}$$

since

$$f(E_{11}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 \\ -3 & 0 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -12 \\ -3 & 0 \end{pmatrix}$$

$$= 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(E_{12}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} -3 & -6 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 12 \\ 0 & -3 \end{pmatrix}$$

$$= 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 12\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(E_{21}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 0 \\ -6 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 6 & 12 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ -12 & -12 \end{pmatrix}$$

$$= 12\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(E_{22}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 12 \\ 0 & -6 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 3 & 0 \end{pmatrix}$$

$$= 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 12\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$A \in M_n(\mathbb{F})$ and

$$\boxed{\begin{aligned} f : V &\longrightarrow V \\ X &\longmapsto AX - XA. \end{aligned}}$$

Recall that

$$\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{F} \text{ is}$$

given by $\langle X, Y \rangle = tr(XY)$.

So

$$\begin{aligned} \langle f(X), Y \rangle &= \langle AX - XA, Y \rangle \\ &= tr((AX - XA)Y) \\ &= tr(AXY - XAY) \\ &= tr(AXY) - tr(XAY) \\ &= tr(XYA) - tr(XAY) \\ &= tr(XYA - XAY) \end{aligned}$$

$= \mathrm{tr}(X(YA - AY))$

$= \langle X, YA - AY \rangle$

$= \langle X, -f(Y) \rangle$

So $\langle f(X), Y \rangle = \langle X, -f(Y) \rangle.$

So $f^* = -f.$