GTLA Lecture 29.10.2020

A $n \times n$ matrix with entries in $\mathbb{F}$ is a function

$$X : \{1,\dots,n\} \times \{1,\dots,n\} \longrightarrow \mathbb{F}$$

$$X = \begin{pmatrix} X(1,1) & \cdots & X(1,n) \\ \vdots & & \vdots \\ X(n,1) & \cdots & X(n,n) \end{pmatrix}$$

Let $V = M_n(\mathbb{F})$. Then V is an $\mathbb{F}$-vector space with

$$V \times V \to V \quad \text{and} \quad \mathbb{F} \times V \longrightarrow V$$
$$(X,Y) \mapsto X+Y \qquad (c,X) \longmapsto cX.$$

where

$$(X+Y)(i,j) = X(i,j) + Y(i,j)$$
$$(cX)(i,j) = c \cdot X(i,j)$$

Then $\dim V = n^2$. Define

$$tr : V \longrightarrow \mathbb{C}$$
$$X \longmapsto \sum_{i=1}^{n} X(i,i).$$

$$tr(X) = X(1,1) + \cdots + X(n,n).$$

<u>Proposition</u>

(a) tr is a linear transformation.

(b) If $X, Y \in V$ then
$$tr(XY) = tr(YX).$$

<u>Proof</u>

(a) To show: (aa) If $X, Y \in V$ then
$$tr(X+Y) = tr(X) + tr(Y).$$
(ab) If $c \in \mathbb{F}$ and $X \in V$ then
$$tr(cX) = c\, tr(X).$$

(aa) Assume $X, Y \in V$.

To show: $tr(X+Y) = tr(X) + tr(Y)$.

$$tr(X+Y) = \sum_{i=1}^{n} (X+Y)(i,i)$$
$$= \sum_{i=1}^{n} X(i,i) + Y(i,i)$$
$$= \sum_{i=1}^{n} X(i,i) + \sum_{i=1}^{n} Y(i,i)$$

$= tr(X) + tr(Y).$

(ab) Assume $c \in \mathbb{F}$ and $X \in V$.

To show: $tr(cX) = c\, tr(X)$.

$$tr(cX) = \sum_{i=1}^{n} (cX)(i,i)$$

$$= \sum_{i=1}^{n} c \cdot X(i,i) = c \sum_{i=1}^{n} X(i,i)$$

$$= c\, tr(X).$$

(b) To show: If $X, Y \in V$ then
$tr(XY) = tr(YX)$.

Assume $X, Y \in V$.

To show: $tr(XY) = tr(YX)$.

$$tr(XY) = \sum_{i=1}^{n} (XY)(i,i)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} X(i,j) Y(j,i) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} Y(j,i) X(i,j).$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} y(j,i)\, x(i,j)$$

$$= \sum_{j=1}^{n} (yx)(j,j) = tr(yx). \quad //$$

Let $n=2$. Then if $\mathbb{F} = \mathbb{C}$ then

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis of V.

and

$C = \{6\}$ is a basis of $\mathbb{C}$.

The matrix of $tr : V \longrightarrow \mathbb{C}$.

with respect to B and C is

$$\boxed{\begin{array}{c} i6 = 6i \\ \left(\frac{1}{6} i\right) \cdot 6 = i. \end{array}}$$

$$A = \left(\frac{1}{6}, 0, \overset{0}{\cancel{\tfrac{1}{6}}}, \overset{\frac{1}{6}}{\cancel{0}} \right)$$

Since

$$\operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = \underline{\underline{\tfrac{1}{6}}} \cdot 6$$

$$\operatorname{tr}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 = \underline{\underline{0}} \cdot 6$$

$$\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 = \underline{\underline{0}} \cdot 6$$

$$\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 = \underline{\underline{\tfrac{1}{6}}} \cdot 6$$

Let $V = M_n(\mathbb{F})$. Define

$\langle\, , \rangle : V \times V \longrightarrow \mathbb{F}$ by

$\langle X, Y \rangle = \operatorname{tr}(XY)$.

<u>Proposition</u> Let $X, Y, Z \in V$ and $c \in \mathbb{F}$. Then

(a) $\langle cX, Y \rangle = c\langle X, Y \rangle$

(b) $\langle Y, X \rangle = \langle X, Y \rangle$.

(c) $\langle X, cY \rangle = c\langle X, Y \rangle$

(d) $\langle X+Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle$

(e) $\langle X, Y+Z \rangle = \langle X, Y \rangle + \langle X, Z \rangle$

Proof Assume $X, Y, Z \in V$ and $c \in \mathbb{F}$.

(a) $\langle cX, Y \rangle = tr(cXY)$

$= c\,tr(XY) = c\langle X, Y \rangle$.

(b) $\langle Y, X \rangle = tr(YX)$

$= tr(XY) = \langle X, Y \rangle$.

(c) $\langle X, cY \rangle = \langle cY, X \rangle$ (by (b)).

$= c\langle Y, X \rangle$ (by (a))

$= c\langle X, Y \rangle$ (by (b)).

(d) $\langle X+Y, Z \rangle = tr((X+Y)Z)$.

$= tr(XZ + YZ)$

$= tr(XZ) + tr(YZ)$

$= \langle X, Z \rangle + \langle Y, Z \rangle$.

(e) $\langle X, Y+Z \rangle = \langle Y+Z, X \rangle$ (by (b)) symmetry

$= \langle Y, X \rangle + \langle Z, X \rangle$ (by (d))

$= \langle x, y \rangle + \langle x, z \rangle$ (by (b)) symmetry

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Proposition

(a) If $X \in V$ and $X \neq 0$ then there exists $Y \in V$ such that $\langle X, Y \rangle \neq 0$ (nondegeneracy).

(b) $V^{\perp} = \{0\}$ and $V \cap V^{\perp} = \{0\}$.

(c) $\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = 0.$

Proof

(c) $\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$

$$= tr\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = tr \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0.$$

So, if $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in V$ then $\langle X, X \rangle = 0$.

So there are vectors of "<u>length 0</u>" (isotropic vectors).

(a) Let E_{ij} be the matrix with 1 in (i,j) entry and 0 elsewhere. For example

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_{11}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E_{12}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_{21}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_{22}$$

("matrix units").

To show: If $x \in V$ and $x \neq 0$ then there exists $y \in V$ such that

$$\langle x, y \rangle \neq 0$$

Assume $x \in V$ and $x \neq 0$.

To show: There exists $y \in V$ with $\langle x, y \rangle \neq 0$

Since $x \neq 0$ then there exist $i, j \in \{1, \dots, n\}$ such that

$$x(i,j) \neq 0.$$

Let

$$y = E_{ji} = \begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ j & \begin{pmatrix} & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & \end{pmatrix} \end{array}$$

To show: $\langle X, Y\rangle \neq 0$.

$$\langle X, Y\rangle = \operatorname{tr}(XY) = \operatorname{tr}(X E_{ji})$$

$$= \operatorname{tr}(X E_{j1} E_{1i})$$

$$\begin{array}{cc} & \begin{array}{cc} & i \end{array} \\ j & \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cc} & i \end{array} \\ j & \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array}$$

$$= \operatorname{tr}(E_{1i} X E_{j1})$$

$$= \operatorname{tr}(E_{1i} (E_{ii} X E_{jj}) E_{j1})$$

$$= \operatorname{tr}(E_{1i} (X(i,j) E_{ij}) E_{j1})$$

$$X(i,j) E_{ij} = X(i,j) \begin{array}{cc} & \begin{array}{cc} & j \end{array} \\ i & \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{c} j \end{array} \\ i & \begin{pmatrix} 0 & \\ & X(i,j) \\ 0 & \end{pmatrix} \end{array}$$

$$= \operatorname{tr}(X(i,j) E_{1i} E_{ij} E_{j1})$$

$$= X(i,j) \operatorname{tr}(E_{1i} E_{ij} E_{j1})$$

$$= X(i,j) \operatorname{tr}(E_{11}) = X(i,j) \cdot 1 = X(i,j) \neq 0.$$

Let $\mathbb{E}^2 = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$

and ~~let~~

$$r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and H the subgroup of $GL_3(\mathbb{R})$ generated by r and s.

Calculate the stabilizer and orbit of $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ under the action of H.

$$\mu : G/N \times G/N \to G/N$$
$$(aN, bN) \to abN$$

is a function

if and only if N is normal.