GTLA Lecture 16.10.2020

Tensor product of vector spaces.

$\dim(V)=m$ $\quad$ $\mathrm{Card}(B)=m$

V has basis B $\quad$ B and C are sets.

$\dim(W)=n$

W has basis C

$\mathrm{Card}(C)=n$

$V \oplus W$ has basis $B \cup C$

$V \otimes W$ has basis $B \times C = \left\{ (b,c) \,\middle|\, \begin{matrix} b \in B \\ c \in C \end{matrix} \right\}$

$\dim(V \otimes W) = mn$. $\quad$ $\mathrm{Card}(B \times C) = mn$.

Let G be a group.

Let $\mathcal{S}$ be the set of subsets of G.

The <u>conjugation action</u> of G on $\mathcal{S}$

$$G \times \mathcal{S} \to \mathcal{S}$$

$$(g, S) \mapsto g \diamond S \quad \text{where}$$

$$g \diamond S = gSg^{-1} = \{ gxg^{-1} \mid x \in S \}$$

<u>Example</u> $D_2 = \{1, r, s, sr\} = G$

with $r^2 = 1$, $s^2 = 1$ and $rs = sr^{-1}$.

$$\mathcal{S} = \left\{ \begin{array}{c} \phi, \\ \{1\}, \quad \{r\}, \quad \{s\}, \quad \{sr\} \\ \{1,r\}, \ \{1,s\}, \ \{1,sr\}, \ \{r,s\}, \ \{r,sr\}, \\ \{1,r,s\}, \ \{1,r,sr\}, \ \{1,s,sr\} \\ \{r,s,sr\} \\ \{1,r,s,sr\} \end{array} \right\}$$

$\mathrm{Card}(\mathcal{S}) = 16 = 2^4.$

$$r \odot \{1,r,s\} = \{r1r^{-1}, rrr^{-1}, rsr^{-1}\}$$

$$= \{1, r, s\}, \text{ since } rsr^{-1} = sr^{-1}r^{-1} = sr^{-2} = s \cdot 1 = s$$

$$r \circlearrowright \{1,r,s\}$$

Let G be a group, $\mathcal{S}$ the set of subsets of G, and G acts on $\mathcal{S}$ by conjugation.

Let $S \in \mathcal{S}$

The <u>normalizer of S</u> in G is

$$N(S) = \{g \in G \mid gSg^{-1} = S\}$$
$$= \{g \in G \mid g \diamond S = S\}$$
$$= \mathrm{Stab}_G(S)$$

(Since $N(S)$ is a stabilizer then $N(S)$ is a subgroup of G).

<u>Proposition</u> Let H be a subgroup of G.

(a) H is a normal subgroup of $N(H)$

$$H \leq N(H) \leq G$$

maybe not normal in G

H <u>is</u> normal in $N(H)$.

(b) If K is a subgroup of G and H is a normal subgroup of K

$$H \leq K \leq G$$

then $K \leq N(H)$

$N(H)$ is the <u>largest</u> subgroup in which H is normal.

<u>Proof</u> (a) To show: H is normal in $N(H)$.

To show: If $h \in H$ and $n \in N(H)$

then $nhn^{-1} \in H$.

Assume $h \in H$ and $n \in N(H)$.

To show: $nhn^{-1} \in H$.

Since $n \in N(H)$ then $nHn^{-1} = H$,

so $nhn^{-1} \in nHn^{-1} = H$.

So $nhn^{-1} \in H$.

Why is $H \subseteq N(H)$? to make H a subgp of $N(H)$.

(b) To show: If K is a subgroup of G and H is normal in K then $K \subseteq N(H)$.

Assume K is a subgroup of G and H is normal in K.

To show: $K \subseteq N(H)$.

To show: If $k \in K$ then $k \in N(H)$.

Assume $k \in K$.

To show: $k \in N(H)$.

To show: $kHk^{-1} = H$.

To show: (ba) $kHk^{-1} \subseteq H$

(bb) $H \subseteq kHk^{-1}$.

(ba) Assume $y \in kHk^{-1}$.

To show: $y \in H$

There exists $h \in H$ s.t. $y = khk^{-1}$.

Since H is normal in K then

$$y = khk^{-1} \in H.$$

So $kHk^{-1} \subseteq H$.

(bb) To show: $H \subseteq kHk^{-1}$

To show: If $h \in H$ then $h \in kHk^{-1}$.

Assume $h \in H$

To show: $h \in kHk^{-1}$.

To show: There exists $h_1 \in H$ such that $h = kh_1k^{-1}$.

Let $h_1 = k^{-1}hk$.

Then

$$k(h_1)k^{-1} = k(k^{-1}hk)k^{-1}$$
$$= kk^{-1}hkk^{-1} = h.$$

So $h = khk^{-1}$.

So $h \in kHk^{-1}$.

So $H \subseteq kHk^{-1}$.

So $H = kHk^{-1}$.

So $k \in N(H)$.

So $K \subseteq N(H)$.

P.

Why is $H \subseteq N(H)$ to make H a subgp of

K is a subgroup normal in $K^{N(H)}$

Since H is normal in H then (6) gives $H \subseteq N(H)$.

The Sylow theorems

Let G be a finite group

Let $p \in \mathbb{Z}_{>0}$ be prime, and $a, b \in \mathbb{Z}_{>0}$ such that

$$\mathrm{Card}(G) = p^a b$$

$84 = 2^a \cdot b$

where b is not divisible by p.

Example $\mathrm{Card}(G) = 84 = 2^2 \cdot 21$.

The set of p-Sylow subgroups of G is

$$\mathcal{B} = \left\{ Q \;\middle|\; \begin{array}{l} Q \text{ is a subgroup and} \\ \mathrm{Card}(Q) = p^a \end{array} \right\}$$

Example: $p=2$

$$\mathcal{B}_p = \{\text{subgroups of } G \text{ with cardinality } 4 = 2^2\}$$

The group G acts on $\mathcal{B}_p$ by conjugation,

$$G \times \mathcal{B}_p \longrightarrow \mathcal{B}_p$$
$$(g, Q) \longmapsto g \diamond Q = gQg^{-1}.$$

Sylow Theorems

(1) $\mathcal{B}_p \neq \emptyset$.

(2) The action G on $\mathcal{B}_p$ by conjugation has only one orbit.

(3) $\mathrm{Card}(\mathcal{B}_p) = 1 \mod p$.

(4) $\mathrm{Card}(\mathcal{B}_p)$ divides $\mathrm{Card}(G)$.

History 1950's R. Brauer.

Can you determine the finite groups with no normal subgroups
simple group.

1960's finding new simple

groups.

1971, Daniel Gorenstein's plan for doing a complete classification.

1981 Gorenstein said "I think its done" we've found them all.

Sam Lyons, $\boxed{\text{Aschbacher}}$, Steve Smith, Robert Wilson finished it after 30 more years.

Example

$\mathrm{Card}(G) = 84 = 2^2 \cdot 3 \cdot 7$.

$\mathcal{B}_7 = \{7\text{-Sylow subgroups}\}$.
$= \{\text{subgroups of size } 7\}$.

Sylow says:

$\mathrm{Card}(\mathcal{B}_7) = 1 \bmod 7$.

So $\mathrm{Card}(\mathcal{B}_7)$ is 1 or $\cancel{8}$ or $\cancel{15}$ or $\cancel{22}$ or $\cancel{29}$ or $\cancel{36}$ or $\cancel{43}$ or $\cancel{50}$ or ...

Sylow says

$\mathrm{Card}(\mathcal{B}_7)$ divides $\mathrm{Card}(G) = 84$

So $\mathrm{Card}(B_7) = 1$.

Sylow says all 7-Sylow subgroups are conjugate.

So If $B_7 = \{Q\}$

then $gQg^{-1} = Q$.

So Q is normal.

$\mathrm{Card}(G) = 84$.

G is <u>NOT</u> simple since Q is normal.

Let A be the matrix of f with respect to the basis B