GTLA Lecture 15.10.2020

Goal: Conjugacy classes of the dihedral group. D_n.

Let $r, s \in M_n(\mathbb{C})$ be given by

$$r(i,j) = \begin{cases} 1, & \text{if } j = i-1 \bmod n \\ 0, & \text{otherwise.} \end{cases}$$

and

$$s(i,j) = \begin{cases} 1, & \text{if } j = n-i \\ 0, & \text{otherwise.} \end{cases}$$

$r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $n = 2$

$r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ if $n = 3$

$r = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ if $n = 4$

and if $n = 5$

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } s = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The *cyclic group* $\mathbb{Z}_n$ is

$$\mathbb{Z}_n = \{1, r, r^2, \dots, r^{n-1}\}.$$

$\mathbb{Z}_n$ is presented by the generator r and relation $r^n = 1$.

The *dihedral group* D_n is

$$D_n = \left\{ \begin{array}{l} 1, r, r^2, \dots, r^{n-1} \\ s, sr, sr^2, \dots, sr^{n-1} \end{array} \right\}.$$

D_n is presented by generators r, s and relations

$$r^n = 1, \quad s^2 = 1, \quad rs = sr^{-1}.$$

The *conjugation action* of D_n on itself is

$$D_n \times D_n \longrightarrow D_n$$
$$(g, x) \longmapsto g \diamond x,$$

where $g \diamond x = gxg^{-1}$.

Explicitly, for $k \in \{0, \dots, n-1\}$.

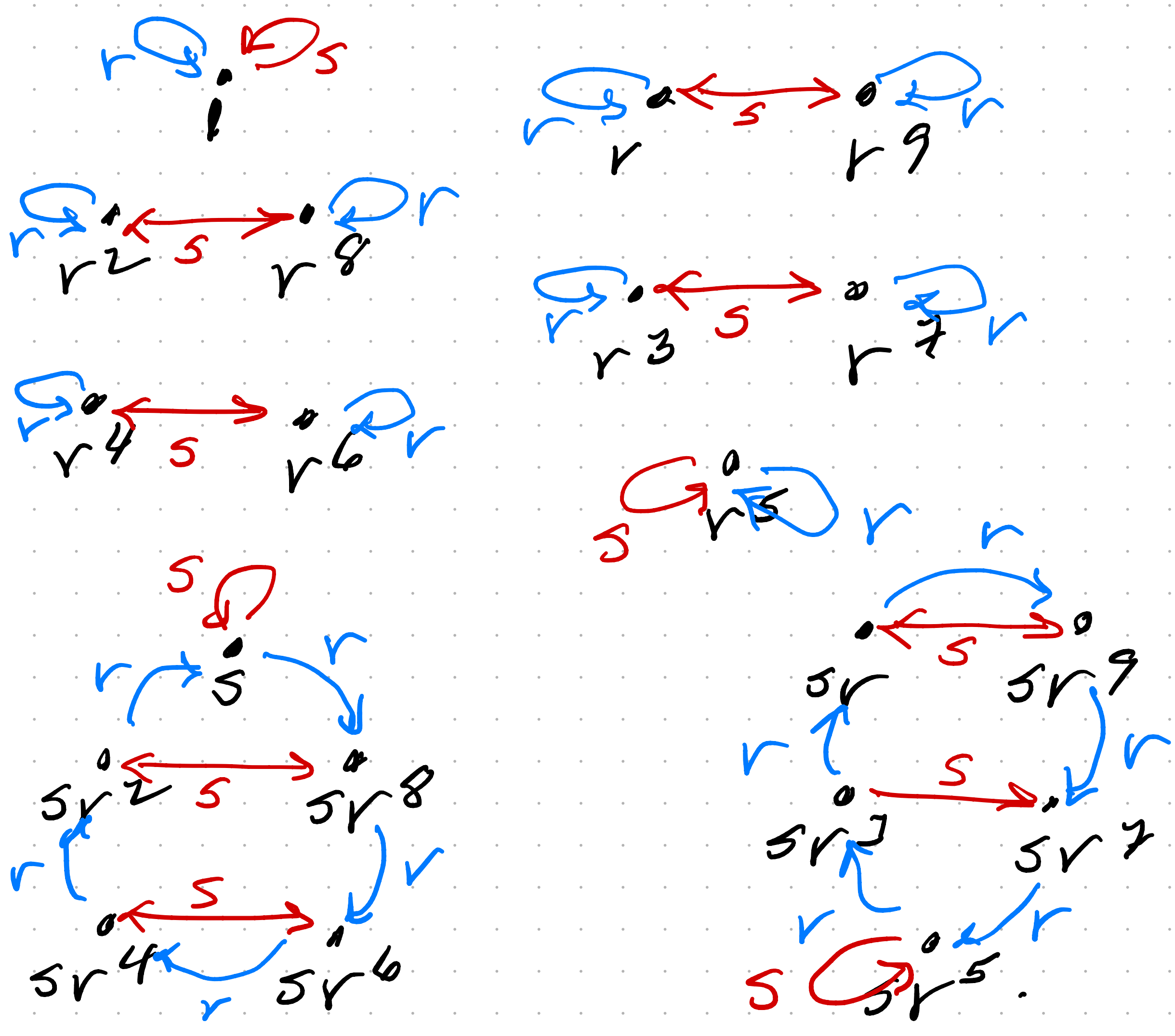
$$r \diamond r^k = r r^k r^{-1} = r^k$$

$$r \diamond sr^k = r(sr^k)r^{-1} = sr^{-1}r^{k-1} = sr^{k-2}$$

$$s \diamond r^k = s r^k s^{-1} = s r^k s = s s r^{-k} = r^{-k} = r^{n-k}$$

$$s \diamond sr^k = s(sr^k)s^{-1} = s s r^k s = r^k s = s r^{-k} = s r^{n-k}.$$

Pictorially for $n = 10$



Conjugacy classes: (orbits)

$\{1\}$

$\{r^2, r^8\}$

$\{r^4, r^6\}$

$\{r, r^9\}$

$\{r^3, r^7\}$

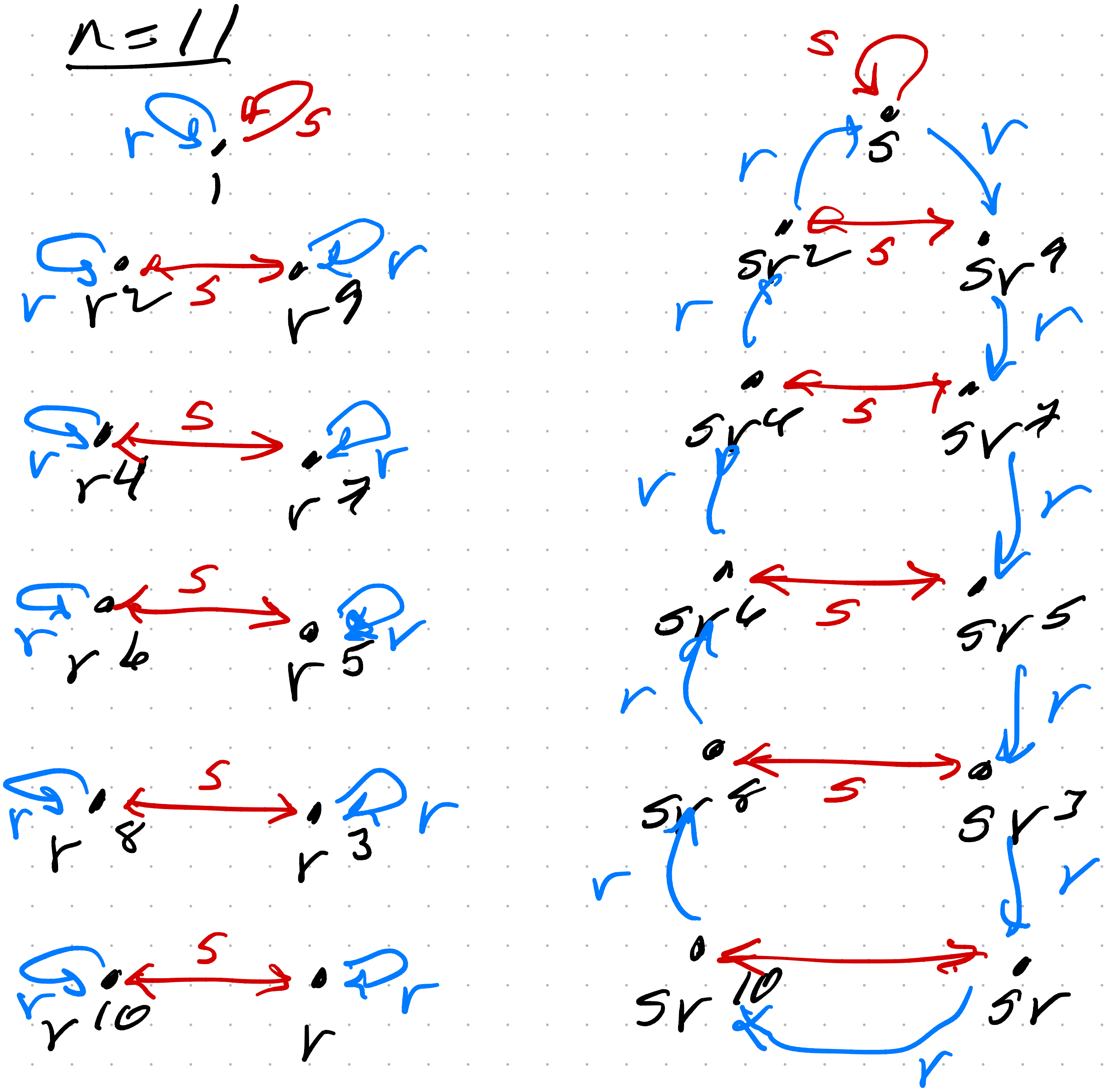
$\{r^5\}$

and

$\{s, sr^2, sr^4, sr^6, sr^8\}$

$\{sr, sr^3, sr^5, sr^7, sr^9\}$

$\underline{n=11}$



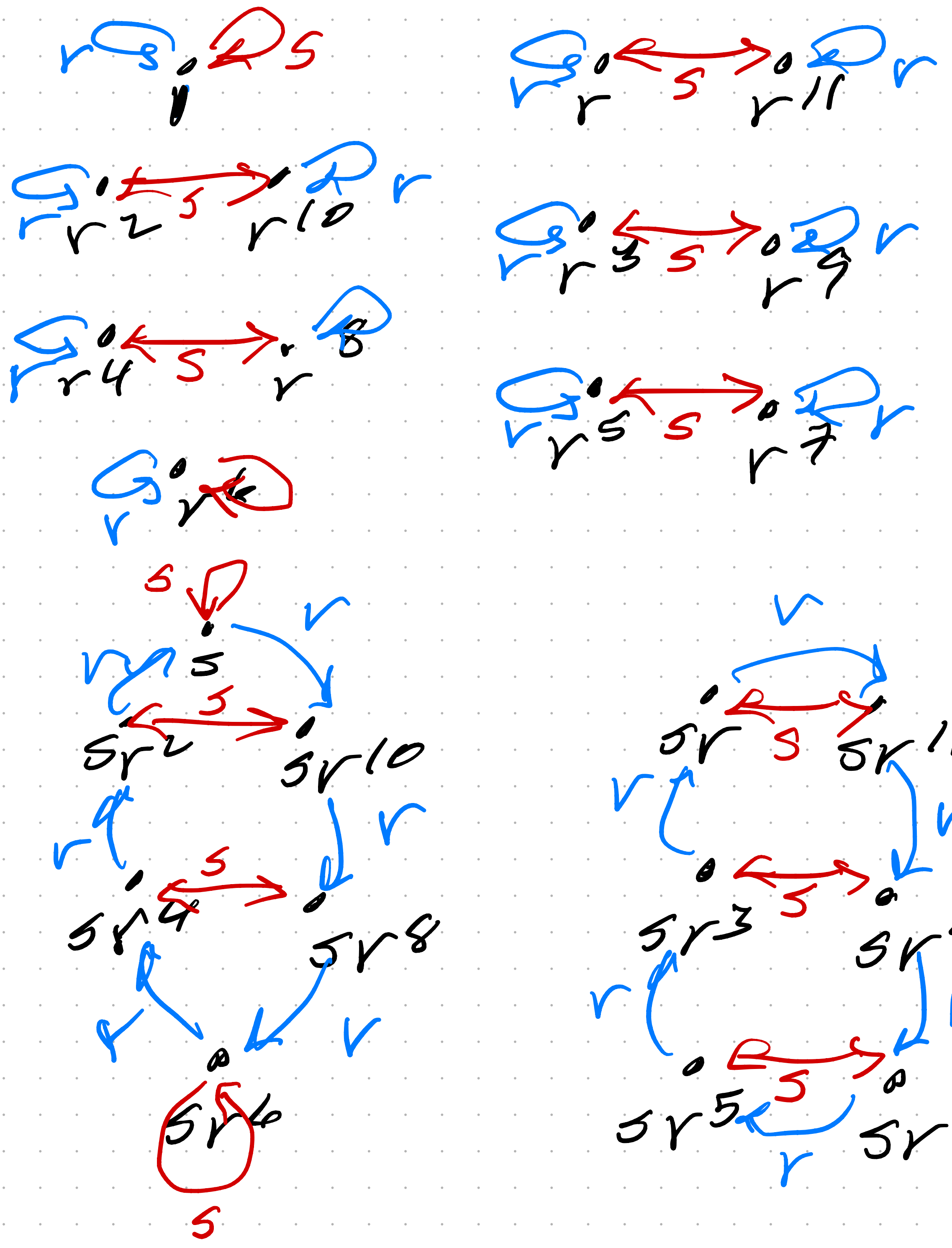
Conjugacy classes

$\{1\}$

$\{r^2, r^9\}$
$\{r^4, r^7\}$
$\{r^6, r^5\}$
$\{r^8, r^3\}$
$\{r^{10}, r\}$

and $\{s, sr^2, sr^4, sr^6, sr^8, sr^{10},$
$sr, sr^3, sr^5, sr^7, sr^9\}$

$n = 12$



Conj. Classes

$\{1\}$
$\{r^2, r^{10}\}$
$\{r^4, r^8\}$
$\{r^6\}$
$\{r, r^{11}\}$
$\{r^3, r^9\}$
$\{r^5, r^7\}$

$\{s, sr^2, sr^4, sr^6, sr^8, sr^{10}\}$

$\{sr, sr^3, sr^5, sr^7, sr^9, sr^{11}\}$.

Orbit counting.

~~Burnside's~~ Lemma (not ~~Burnside's~~ Lemma).

(Fixed points).

<u>Proposition</u> Let S be a finite set. Let G be a group acting on S. Then

$$\begin{pmatrix}\#\text{ of}\\ \text{orbits}\end{pmatrix} \cdot (\mathrm{Card}(G)) = \sum_{g \in G} \begin{pmatrix}\#\text{ of fixed points}\\ \text{of } g\end{pmatrix}.$$

<u>Proof</u>

$$\sum_{g \in G} \begin{pmatrix}\#\text{ of fixed points}\\ \text{of } g\end{pmatrix}$$

$$= \sum_{g \in G} \mathrm{Card}\{x \in S \mid gx = x\}.$$

$$= \mathrm{Card}\{(g,x) \mid g \in G,\ x \in S \text{ and } gx = x\}.$$

$$= \sum_{x \in S} \mathrm{Card}\{g \in G \mid gx = x\}.$$

$$= \sum_{x \in S} \mathrm{Card}(\mathrm{Stab}_G(x))$$

Let $Gx_1, Gx_2, \ldots, Gx_N$ be the distinct orbits.

So $\displaystyle\sum_{x \in S} \mathrm{Card}(\mathrm{Stab}_G(x))$

$$= \sum_{i=1}^{N} \sum_{x \in Gx_i} \mathrm{Card}(\mathrm{Stab}_G(x)).$$

$$= \sum_{i=1}^{N} \sum_{x \in Gx_i} \mathrm{Card}(\mathrm{Stab}_G(x_i)).$$

recall $\mathrm{Stab}_G(gx_i) = g\,\mathrm{Stab}_G(x_i)\,g^{-1}$

so $\mathrm{Card}(\mathrm{Stab}_G(gx_i)) = \mathrm{Card}(\mathrm{Stab}_G(x_i))$.

$$= \sum_{i=1}^{N} \mathrm{Card}(\mathrm{Stab}_G(x_i)) \cdot \begin{pmatrix} \#\text{ of } x \\ \text{in } Gx_i \end{pmatrix}$$

$$= \sum_{i=1}^{N} \mathrm{Card}(\mathrm{Stab}_G(x_i))\mathrm{Card}(Gx_i)$$

$$= \sum_{i=1}^{N} \mathrm{Card}(G) = N\,\mathrm{Card}(G)$$

$$= \begin{pmatrix} \#\text{ of} \\ \text{orbits} \end{pmatrix} \cdot \mathrm{Card}(G) \quad //.$$

$GL_n(\mathbb{C})$.

$$\boxed{\text{A \underline{fixed point} of } g \text{ is } x \in S \text{ such that } gx = x.}$$

If g is a random elt
of $\cancel{GL_n}(\mathbb{C})$ $M_n(\mathbb{C})$
what do its eigenvalues
look like?