GTLA Lecture 13.10.2020

Let $V = \mathbb{C}^n$ with the standard dot product

$$\left\langle \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right\rangle = u_1\bar{v}_1 + u_2\bar{v}_2 + \cdots + u_n\bar{v}_n.$$

Let $f : V \to V$ be a linear transf.

$f^* : V \to V$ the adjoint linear transformation

i.e. if $u, v \in V$ then

$$\langle f(u), v\rangle = \langle u, f^*(v)\rangle$$

Then

(a) f is selfadjoint if $f = f^*$.

(b) f is unitary if $ff^* = 1 = f^*f$.

(c) f is normal if $ff^* = f^*f$.

Let A be the matrix of f with respect to the standard basis, $\{e_1, \ldots, e_n\}$ where

$e_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith. So $f : \mathbb{C}^n \to \mathbb{C}^n$

$v \mapsto Av.$

Then the matrix of $f^* : \mathbb{C}^n \to \mathbb{C}^n$ is $A^* = \bar{A}^t$. (w.r.t. $\{e_1, \dots, e_n\}$)

Then

(1) A is self adjoint, or Hermitian if $A = A^*$ (symmetric if $A \in M_n(\mathbb{R})$).

(2) A is unitary if $AA^* = 1 = A^*A$ (orthogonal if $A \in M_n(\mathbb{R})$).

(3) A is normal if $AA^* = A^*A$.

The "reason" for unitary matrices

Proposition $U_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A \text{ is unitary}\}$

$\{\text{orthonormal bases in } \mathbb{C}^n\} \longleftrightarrow U_n(\mathbb{C})$

$(u_1, \dots, u_n) \longmapsfrom \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$

The reason for normal matrices

<u>Proposition</u> Let $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$.
Let $\lambda \in \mathbb{C}$ and $V_\lambda = \ker(\lambda - A)$ (the λ-eigenspace of A — all eigenvectors of eigenvalue λ).

V_λ is A-invariant, and
V_λ is A^*-invariant, and
$V_\lambda^\perp$ is A-invariant, and
$V_\lambda^\perp$ is A^*-invariant.

<u>Theorem</u> (Spectral theorem)
Let $n \in \mathbb{Z}_{>0}$ and let $V = \mathbb{C}^n$ with standard dot product.

(a) Let $A \in M_n(\mathbb{C})$ such that
$$AA^* = A^*A.$$

Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and a unitary matrix $U \in U_n(\mathbb{C})$ such that

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

(b) Let $f: V \to V$ be a linear transformation which is normal. Then there exist an orthonormal basis of V consisting of eigenvectors of f.

<u>Proof</u> Proof by induction on n.

<u>Base case</u>: $n=1$. Then $A = (\lambda)$ and $U = (1)$ and A is diagonal since it is 1×1.

<u>Induction step</u> $A \in M_n(\mathbb{C})$.

Let $c_A(x) = \det(x - A)$ be the

characteristic polynomial.

Let $\lambda \in \mathbb{C}$ be a root of $c_A(x)$. $\mathbb{C}$ is alg. closed.

So $\det(\lambda - A) = 0$.

So $\ker(\lambda - A) \neq 0$.

So $V_\lambda \neq 0$.

Using Gram-Schmidt produce an orthonormal basis $(u_1, \ldots, u_k)$ of $V_\lambda = \ker(\lambda - A)$. $(\dim V_\lambda = k)$

Note that $u_1, \ldots, u_k \in V_\lambda$ are all eigenvectors of A. (of eigenvalue λ).

(since $V \cap V^\perp = 0$ because we are using std. dot product)

Then $V = V_\lambda \oplus V_\lambda^\perp$ and

Since A is normal then

V_λ is a A-invariant, and A^*-invt

$V_\lambda^\perp$ is A-invt and A^*-invt.

Then

$g : V_\lambda^\perp \to V_\lambda^\perp$ and $g^* : V_\lambda^\perp \to V_\lambda^\perp$
$v \mapsto Av$. $\qquad v \mapsto A^* v$.

satisfy $gg^* = g^* g$ (since $AA^* = A^* A$).

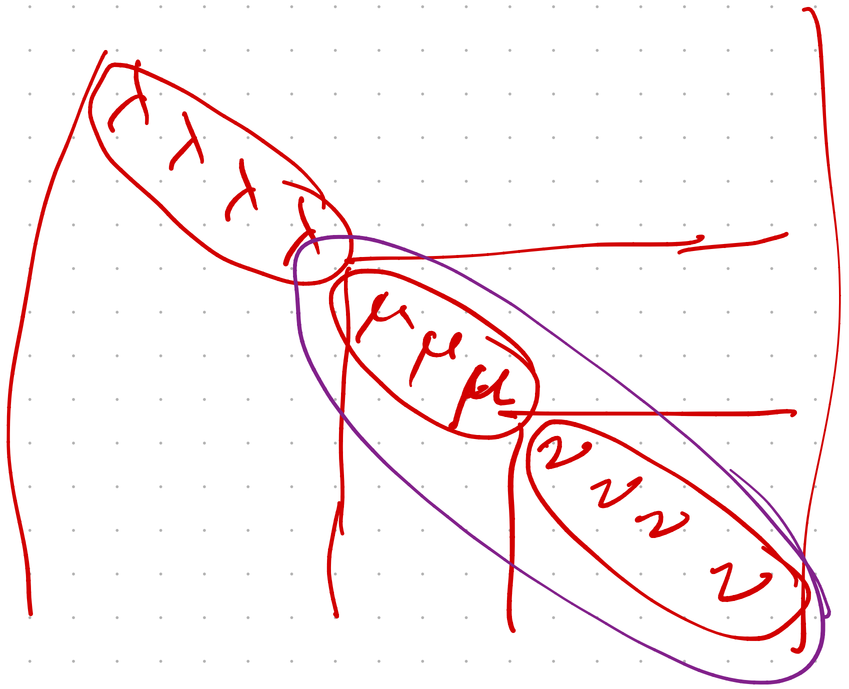
$\dim(V_\lambda^\perp) < n$ since $V_\lambda \neq 0$.
(so $\dim V_\lambda > 0$).

So, <u>by induction</u> there exists an orthonormal basis $(u_{k+1}, \dots, u_n)$ of $V_\lambda^\perp$ consisting of eigenvectors of g. (which are eigenvectors of A).

Also $(u_{k+1}, \dots, u_n)$ are orthogonal to $(u_1, \dots, u_k)$ (since $u_1, \dots u_k \in V_\lambda$ and $u_{k+1}, \dots, u_n \in V_\lambda^\perp$)

Then $(u_1, \dots, u_k, u_{k+1}, \dots, u_n)$ is an orthonormal basis of V.

consisting of eigenvectors of A //.



If $V_1^{\perp} = 0$ then

$$V = V_\lambda$$

Corollary Assume $A \in M_n(\mathbb{C})$ is unitary. Then all eigenvalues of A have absolute value 1.

Proof By the spectral theorem there exists $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $U \in U_n(\mathbb{C})$ such that

$$D = U^{-1}AU = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\begin{aligned} DD^* &= (U^{-1}AU)(U^{-1}AU)^* \\ &= U^{-1}AUU^*A^*(U^{-1})^* \\ &= U^{-1}AUU^{-1}A^*U \\ &= U^{-1}AA^*U \\ &= U^{-1}\cdot 1\cdot U = 1. \end{aligned}$$

So

$$\begin{aligned} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = DD^* &= D\overline{D}^t \\ &= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} \overline{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda}_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1\overline{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\overline{\lambda}_n \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix} \end{aligned}$$

So if $i \in \{1, \dots, n\}$

$|\lambda_i|^2 = 1$. $\quad (|\lambda_i|^2 \in \mathbb{R}_{\geq 0})$.

So $|\lambda_i| = 1$. //

$\begin{pmatrix} | \\ u_i \\ | \end{pmatrix}$ and $\begin{pmatrix} | \\ u_j \\ | \end{pmatrix}$ <u>orthogonal</u> means

$$0 = \begin{pmatrix} u_{1i} \\ \vdots \\ u_{ni} \end{pmatrix} \cdot \begin{pmatrix} u_{1j} \\ \vdots \\ u_{nj} \end{pmatrix} = u_{1i}\bar{u}_{1j} + \cdots + u_{ni}\bar{u}_{nj}$$

$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$ is <u>unitary</u> means

$$1 = U^* U = \bar{U}^t U$$

$$= \begin{pmatrix} — \bar{u}_1 — \\ — \bar{u}_2 — \\ \vdots \\ — \bar{u}_n — \end{pmatrix} \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} & & \\ & u_i \cdot u_j & \\ & & \end{pmatrix}$$

$f: \mathbb{F}^n \to \mathbb{F}^n \longleftrightarrow$? If $A \in M_n(\mathbb{F})$.

$$\underline{\underline{A^* = \bar{A}^t}}$$

Yes standard.