GTLA Lecture 02.10.2020

Let $\langle , \rangle : V \times V \to \mathbb{F}$ a sesquilinear form and W is a finite dim'l subspace of V and $W \cap W^{\perp} = 0$.

Let $f : W \to W$ be a linear transf. and $f^* : W \to W$ the adjoint linear transformation.

(if $x, y \in W$ then $\langle f(x), y \rangle = \langle x, f^*(y) \rangle$)

(a) f is <u>self-adjoint</u> if $f = f^*$.

(b) f is an <u>isometry</u> if $ff^* = 1 = f^*f$

(c) f is <u>normal</u> if $f^*f = ff^*$.

Favourite example is when $V = \mathbb{C}^n$ and $\langle , \rangle$ is the standard dot product.

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 \overline{v_1} + u_2 \overline{v_2} + \cdots + u_n \overline{v_n}$$

Let A be the matrix of f

and A^* be the matrix of f^*

with respect to the favorite basis $\{e_1, \dots, e_n\}$ where $e_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith

$$f : \mathbb{C}^n \to \mathbb{C}^n \qquad v \mapsto Av$$ and $$f^* : \mathbb{C}^n \to \mathbb{C}^n \qquad v \mapsto A^* v.$$

and
$$A^* = \bar{A}^t.$$ (conjugate transpose = Hermitian dual).

(Note: $(AB)^* = \overline{(AB)^t} = \overline{B^t A^t}$
$= \bar{B}^t \bar{A}^t = B^* A^*$)

A is <u>Hermitian</u> if $A = A^*$
(i.e. self-adjoint).

A is <u>unitary</u> if $AA^* = 1 = A^*A$
(i.e. isometry).

A is <u>normal</u> if $AA^* = A^*A$.

What is reason for unitary matrices?

The <u>general linear group</u> is

$GL_n(\mathbb{C}) = \{ P \in M_n(\mathbb{C}) \mid P \text{ is invertible} \}$.

Proposition The map

$$\left\{ \begin{array}{l} \text{ordered bases} \\ (p_1, \dots, p_n) \text{ of } \mathbb{C}^n \end{array} \right\} \longleftrightarrow GL_n(\mathbb{C})$$

$$(p_1, \dots, p_n) \longmapsto \begin{pmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{pmatrix}$$

is a bijection

The unitary group is

$U_n(\mathbb{C}) = \{ U \in M_n(\mathbb{C}) \mid U \text{ is unitary} \}$

($U_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$).

Proposition The map

$$\left\{ \begin{array}{l} \text{ordered} \\ \text{orthonormal} \\ \text{bases in } \mathbb{C}^n \end{array} \right\} \longleftrightarrow U_n(\mathbb{C})$$

$$(u_1, \dots, u_n) \longmapsto \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$$

What is the reason for normal matrices?

A subspace W is <u>A-invariant</u> if W satisfies:

if $w \in W$ and $Aw \in W$.

<u>Proposition</u> Let $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$. Let $\lambda \in \mathbb{C}$ and

$$\begin{aligned} V_\lambda &= \ker(\lambda - A) \\ &= \{v \in \mathbb{C}^n \mid (\lambda - A)v = 0\} \\ &= \{p \in \mathbb{C}^n \mid Ap = \lambda p\}. \end{aligned}$$

(λ-eigenspace of A).

Then

(a) V_λ is A-invariant

(b) V_λ is A^*-invariant

(c) $V_\lambda^\perp$ is A-invariant <u>and</u>

(d) $V_\lambda^\perp$ is A^*-invariant

<u>Proof</u> (a) To show: If $p \in V_\lambda$ then $Ap \in V_\lambda$.

Assume $p \in V_\lambda$.

To show: $Ap \in V_\lambda$.

$Ap = \lambda p \in V_\lambda$ since V_λ is a subspace and $\lambda \in \mathbb{C}$.

(b) To show: If $p \in V_\lambda$ then $A^* p \in V_\lambda$.

Assume $p \in V_\lambda$.

To show: $A^* p \in V_\lambda$.

To show: $A(A^* p) = \lambda (A^* p)$.

$AA^* p = A^* A p$ (since A is normal)

$= A^* \lambda p = \lambda A^* p$.

So V_λ is A^*-invariant.

(d) To show: $V_\lambda^\perp$ is A^*-invariant.

To show: If $v \in V_\lambda^\perp$ then $A^* v \in V_\lambda^\perp$.

Assume $v \in V_\lambda^\perp$.

To show: $A^*v \in V_\lambda^\perp$.

To show: If $p \in V_\lambda$ then $\langle A^*v, p\rangle = 0$

Assume $p \in V_\lambda$.

To show: $\langle A^*v, p\rangle = 0$.

$$\begin{aligned}
\langle A^*v, p\rangle &= \overline{\langle p, A^*v\rangle} \\
&= \overline{\langle Ap, v\rangle} \\
&= \overline{\langle \lambda p, v\rangle} \\
&= \overline{\lambda\langle p, v\rangle} \\
&= \overline{\lambda \cdot 0}, \quad \text{since } v \in V_\lambda^\perp, \\
&= 0.
\end{aligned}$$

So $A^*v \in V_\lambda^\perp$ and $V_\lambda^\perp$ is A^*-invt.

(c) To show: $V_\lambda^\perp$ is A-invariant.

To show: If $v \in V_\lambda^\perp$ then $Av \in V_\lambda^\perp$.

Assume $v \in V_\lambda^\perp$

To show: $Av \in V_\lambda^\perp$.

To show: If $p \in V_\lambda$ then
$$\langle Av, p \rangle = 0.$$
Assume $p \in V_\lambda$.

To show: $\langle Av, p \rangle = 0$.

$$\langle Av, p \rangle = \langle v, A^* p \rangle$$

since $A^* p \in V_\lambda$ since V_λ is A^*-invt.

$$= 0,$$

and $v \in V_\lambda^\perp$.

So $V_\lambda^\perp$ is A-invariant. //

<u>Theorem</u> (Spectral theorem).

Let $n \in \mathbb{Z}_{>0}$ and let $A \in M_n(\mathbb{C})$ and assume $AA^* = A^*A$.

(a) Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and a unitary matrix U such that
$$U^{-1} A U = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

(b) Let $V = \mathbb{C}^n$ with the standard dot product. Let $f: V \to V$ be a linear transformation such that $ff^* = f^*f$. Then there exists an orthonormal basis $\{u_1, \dots, u_n\}$ of V such that $u_1, \dots, u_n$ are all eigenvectors of f.

(a) and (b) are equivalent by taking A to be the matrix of f with respect to $(e_1, \dots, e_n)$ and

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$$

Corollary Assume A is self adjoint. Then A diagonalisable and all eigenvalues are real.

Proof Assume A is self adjoint.

So $A = A^*$.

So $AA^* = AA = A^*A$.

So A is normal.

So the spectral theorem says there exists $U \in U_n(\mathbb{C})$ with

$$U^*AU = U^{-1}AU = \begin{pmatrix} \lambda_1 & & D \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = D$$

Then

$$\begin{pmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{pmatrix} = D^* = (U^*AU)^*$$

$$= U^* A^* (U^*)^*$$

$$= U^*A^*U = U^*AU = D.$$

$$= \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

So $\bar{\lambda}_1 = \lambda_1, \dots, \bar{\lambda}_n = \lambda_n$.

So $\lambda_1 \in \mathbb{R}, \dots, \lambda_n \in \mathbb{R}$. //

group

$$G \hookrightarrow M_n(\mathbb{C})$$

Represent your group as matrices.

$$\begin{pmatrix} 0 & 0 & & \\ \cancel{4} & 1 & & \\ 0 & & & 9 \\ 0 & & \ddots & \end{pmatrix}$$

operation.

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 7 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc|cc} 2 & 3 & 0 & 0 & 14 & 21 \\ 4 & 5 & 0 & 0 & 28 & 35 \\ \hline 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & 5 \\ \hline 4 & 6 & 0 & 0 & 0 & 0 \\ 8 & 10 & 0 & 0 & 0 & 0 \end{array}\right)$$