GTLA Lecture 18.09.2020

Orbit-Stabilizer theorem

"Card(G) = Card(orbit) Card(Stab)."

Let G be a group.

A G-set is a set S with an action of G. An action of G is a function

$$G \times S \longrightarrow S$$
$$(g, x) \longmapsto g \cdot x$$

such that

(a) If $g_1, g_2 \in G$ and $x \in S$ then

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

(b) If $x \in S$ and $1 \cdot x = x$.

Let S be a G-set.

Let $x \in S$.

$$\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$$

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

Theorem (Orbit-Stabilizer Full version).

(a) The orbits partition S.

$$\mathrm{Card}(S) = \sum_{\substack{\text{distinct}\\ \text{orbits}}} \mathrm{Card}(G \cdot x_i).$$

(b) Let $x \in S$ and $H = \mathrm{Stab}_G(x)$.

$$\mathrm{Card}(G/H) = \mathrm{Card}(G \cdot x).$$

(c) Let $x \in S$.

$$\mathrm{Card}(G) = \mathrm{Card}(G \cdot x)\,\mathrm{Card}(\mathrm{Stab}_G(x)).$$

(d) Let $x \in S$ and $g \in G$. Then

$$\mathrm{Stab}_G(g \cdot x) = g\,\mathrm{Stab}_G(x)\,g^{-1}$$

Part (a) We proved last week.

(b) Assume $x \in S$ and let $H = \mathrm{Stab}_G(x)$.

To show: $\mathrm{Card}(G/H) = \mathrm{Card}(G \cdot x)$.

To show: There exists a bijection

$\varphi : G/H \to G\cdot x$

Let $\varphi : G/H \to G\cdot x$
$gH \mapsto g\cdot x$

To show: (a) φ is a function.
(b) φ injective
(c) φ surjective.

(a) To show: If $g_1, g_2 \in G$ and $g_1H = g_2H$ then $\varphi(g_1H) = \varphi(g_2H)$.

Assume $g_1, g_2 \in G$ and $g_1H = g_2H$.

Then $g_1 = g_1 \cdot 1 \in g_1H = g_2H$

so there exists $h \in H$ such that
$$g_1 = g_2h.$$

To show: $\varphi(g_1H) = \varphi(g_2H)$.

To show: $g_1\cdot x = g_2\cdot x$.

$$g_1\cdot x = g_2h\cdot x$$
$$= g_2\cdot(h\cdot x)$$

$= g_2 \cdot x$ (since $h \in H = \mathrm{Stab}_G(x)$).

So φ is a function.

(bb) To show: φ is injective.

To show: If $g_1, g_2 \in G$ and $\varphi(g_1 H) = \varphi(g_2 H)$ then

$$g_1 H = g_2 H.$$

Assume $g_1, g_2 \in G$ and $\varphi(g_1 H) = \varphi(g_2 H)$.

Then $g_1 \cdot x = g_2 \cdot x$.

So $x = g_1^{-1} \cdot g_2 \cdot x = g_1^{-1} g_2 \cdot x$.

So $g_1^{-1} g_2 \in \mathrm{Stab}_G(x) = H$.

To show: $g_1 H = g_2 H$.

$$g_2 = g_1 g_1^{-1} g_2 = g_1 (g_1^{-1} g_2) \in g_1 H.$$

So $g_2 H \cap g_1 H \neq \emptyset$

So $g_1 H = g_2 H$. (since cosets partition G).

So φ is injective.

(bc) To show: φ is surjective.

To show: If $y \in G \cdot x$ then there exist $g \in G$ such that

$$\varphi(gH) = y$$

Assume $y \in G \cdot x$.

Then there exists $g \in G$ such that

$$y = g \cdot x$$

Then

$$\varphi(gH) = g \circ x = y.$$

So φ is surjective.

So $\mathrm{Card}(G/H) = \mathrm{Card}(G \cdot x)$.

(c) Let $x \in S$.

To show: $\mathrm{Card}(G) = \mathrm{Card}(G \cdot x) \cdot \mathrm{Card}(\mathrm{Stab}_G(x))$

Let $H = \mathrm{Stab}_G(x)$.

RHS $= \mathrm{Card}(G \cdot x) \mathrm{Card}(\mathrm{Stab}_G(x))$

$= \mathrm{Card}(G/H)\,\mathrm{Card}(\mathrm{Stab}_G(x))$

(by (b)).

$= \mathrm{Card}(G/H)\,\mathrm{Card}(H)$

$= \mathrm{Card}(G)$ (by Lagrange's theorem).

(d) Assume $x \in S$ and $g \in G$.

To show: $\mathrm{Stab}_G(g \cdot x) = g\,\mathrm{Stab}_G(x)\,g^{-1}$.

To show: (da) $\mathrm{Stab}_G(g\cdot x) \subseteq g\,\mathrm{Stab}_G(x)\,g^{-1}$

(db) $g\,\mathrm{Stab}_G(x)\,g^{-1} \subseteq \mathrm{Stab}_G(g\cdot x)$.

(da) Let $y \in \mathrm{Stab}_G(g\cdot x)$.

To show: $y \in g\,\mathrm{Stab}_G(x)\,g^{-1}$.

To show: There exists $h \in \mathrm{Stab}_G(x)$ such that $y = ghg^{-1}$.

Let $h = g^{-1}yg$.

Then

$h\cdot x = g^{-1}yg\cdot x = g^{-1}\cdot(y\cdot(g\cdot x))$

$= g^{-1} \cdot (g \cdot x)$

$= g^{-1}g \cdot x = 1 \cdot x = x.$

So $h \in Stab_G(x)$.

To show: $y = ghg^{-1}$.

~~$y = g(g^{-1}yg)g^{-1} = \ldots$~~

$=$

$ghg^{-1} = g(g^{-1}yg)g^{-1} = gg^{-1}ygg^{-1}$

$= 1 \cdot y \cdot 1 = y.$

(db) To show: $g\, Stab_G(x) g^{-1}$

$\subseteq Stab_G(g \cdot x)$.

Let $z \in g\, Stab_G(x) g^{-1}$.

Then there exists $h \in Stab_G(x)$ such that $z = ghg^{-1}$.

To show: $z \in Stab_G(g \cdot x)$.

$z \cdot (g \cdot x) = ghg^{-1} \cdot (g \cdot x)$.

$= ghg^{-1}g \cdot x$

$$= gh \cdot x$$

$$= g \cdot (h \cdot x)$$

$$= g \cdot x \quad \left(\text{since } h \in Stab_G(x)\right).$$

So $z \in Stab_G(g \cdot x)$.

So $g\, Stab_G(x) g^{-1} = Stab_G(g \cdot x)$.

Conjugation action and Class equation

$H = Stab_G(x)$

$$G \cdot x = \left(\bigcup_{\text{cosets}} gH\right) \cdot x$$

$$= \bigcup_{\text{cosets}} g(H \cdot x).$$

$$= \underbrace{\bigcup_{\text{cosets}} g \cdot x}_{\text{\# elts is \# of cosets}}.$$

1 elt for each coset in G/H.

<u>Yesterday</u> $f: G \to H$ a homomorphism and $K = \ker f$.

Then $G/K \cong \operatorname{im} f$ as groups

~~Today~~ S is a G-set, $x \in S$ and $H = \operatorname{Stab}_G(x)$.

$$G/H \cong G \cdot x \quad \text{as sets}$$

<u>Category</u>	<u>Morphisms</u>
Sets	functions
Groups	homomorphisms
Vector spaces	Linear transformation.

Rank/nullity theorem.