Tutorial 6

Main topics: Properties and examples of groups; subgroups, cyclic groups, orders of elements

1. Let K be a field.
   (a) Does the set $M_n(K)$ of $n \times n$ matrices form a group under matrix addition?
   (b) Does $M_n(K)$ form a group under matrix multiplication?
   (c) The set $\mathrm{GL}_n(K)$ of invertible $n \times n$ matrices forms a group under matrix multiplication. Does $\mathrm{GL}_n(K)$ form a group under matrix addition?

2. Let G be a group and $x, y, z, w \in G$.
   (a) Given that $xyz^{-1}w = e$, solve for y (in terms of x, z and w).
   (b) Assume $xyz = e$. Does it follow that $yzx = e$? Does it follow that $yxz = e$?

3. Let $n \in \mathbb{N}$. Show that the set of all complex n-th roots of unity $\mu_n = \{z \in \mathbb{C} \mid z^n = 1\}$ forms a group under multiplication. (It's useful to notice that μ_n is a subset of the group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, so it is enough to show that μ_n is a subgroup.)

4. Compute the following products of permutations in the symmetric group S_6:
   (a) $(123)(456) \times (12)(34)(56)$ (b) $(12) \times (246) \times (123654)$

5. (a) For each of the following $n \in \mathbb{N}$ write down an element of S_5 that has order n.
   (i) 1 (ii) 2 (iii) 3 (iv) 4 (v) 5 (vi) 6
   (b) Find the orders of the following elements of the group $\mathbb{C}^\times = (\mathbb{C} \setminus \{0\}, \times)$.
   (i) 1 (ii) -1 (iii) 3 (iv) i

6. List all the cyclic subgroups of S_3. How many are there?

7. Show that if $g^2 = e$ for all g in a group G, then G is abelian.