Tutorial 4

Main topics: Minimal polynomials, diagonalisation

1. Show that the matrix
$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$
has minimal polynomial $X^2 - 2X - 8$. Use this to determine the inverse of A.

2. Find the minimal polynomials of the matrices:
$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$
[Hint: Use the following results from the lectures:
(i) the minimal polynomial divides the characteristic polynomial,
(ii) every eigenvalue is a root of the minimal polynomial.]

3. Let $f\colon V \to V$ be a linear transformation on an n-dimensional vector space with minimal polynomial $m(X) = X^n$.
   (a) Show that there is a vector $v \in V$ such that $f^{n-1}(v) \neq 0$.
   (b) Show that $\mathcal{B} = \{f^{n-1}(v), f^{n-2}(v), \dots, f^2(v), f(v), v\}$ is a basis for V.
   (c) Find the matrix of f with respect to the basis $\mathcal{B}$.

4. Let K be an algebraically closed field, let $V = M_2(K)$ and consider the linear transformation $f\colon V \to V$ given by $f(A) = A^T$ (the transpose of A).
   (a) Find the minimal polynomial of f.
   (b) Find the eigenvalues and corresponding eigenspaces of f.

5. Let $f\colon V \to V$ be a linear transformation on an n-dimensional complex vector space satisfying $f^4 = \mathrm{id}_V$. Show that f is diagonalisable. What can you say about the eigenvalues of f?

6. Let $f\colon V \to V$ be a linear transformation on a finite-dimensional vector space V and suppose that $f^2 = f$. Explain how to find a diagonal matrix representing f.

7. Let $f, g\colon V \to V$ be commuting linear transformations: $fg = gf$.
   (a) Show that any eigenspace of f is g-invariant.
   (b) If K is algebraically closed and V is finite-dimensional, prove that f and g have a common eigenvector.

8. (Challenging!) Find a linear transformation $f\colon V \to V$ on an infinite dimensional vector space V that satisfies no monic polynomial equation $p(f) = 0$.