MAST20022 Group Theory and Linear Algebra

Sample exam 2

Question A1.

(a) Use the Euclidean algorithm to compute $\gcd(54, 42)$.

(b) Are there integers x and y such that $54x + 42y = 3$? If yes, find such integers. If no, explain why not.

Question A2.

(a) List all squares in $\mathbb{F}_7$, that is, all congruence classes $[a]_7$ such that there exists a class $[b]_7$ with $[a]_7 = [b^2]_7$.

(b) Show that $\mathbb{F}_7$ is not algebraically closed.

Question A3. Let $f\colon \mathbb{C}^8 \to \mathbb{C}^8$ be a linear transformation with characteristic polynomial $c(x) = x(x-1)$ and minimal polynomial $x(x-1)^2$. Determine all possible Jordan normal forms of f.

Question A4. Let V be an inner product space.

(a) Define "isometry on V".

(b) Let f be an isometry of V. Show that the eigenvalues of V must have absolute value 1.

Question A5. For each of the given pairs of groups, determine if they are isomorphic or not and briefly justify your answer.

(a) $\mathbb{Z}/30\mathbb{Z}$ and $S_3 \times \mathbb{Z}/5\mathbb{Z}$.

(b) The group of non-zero real numbers under multiplication, and the group of non-zero complex numbers under multiplication.

(c) $GL_2(\mathbb{F}_3)/SL_2(\mathbb{F}_3)$ and $\mathbb{Z}/2\mathbb{Z}$.

Question A6.

(a) Give an example of a group G and a injective group homomorphism $G \to G$ that is not surjective.

(b) Give an example of a group G and a surjective group homomorphism $G \to G$ that is not injective.

Question A7.

(a) Define "normal subgroup".

(b) Show that if H is a subgroup of G of index 2, then H is normal in G.

Question A8. Let V be an inner product space, and $f\colon V \to V$ a self-adjoint linear transformation.

(a) Show that the kernel $K = ker(f)$ of f and image $I = \mathrm{im}(f)$ of f are orthogonal, that is, $I \subseteq K^{\perp}$.

(b) Show that if V is finite-dimensional, then $I = K^{\perp}$.

Question A9. Consider the action of S_4 on the set $\{1, 2, 3, 4\}$ via permutation.

(a) Describe the orbit and stabiliser of 4.

(b) State the orbit-stabiliser relation and verify it in this case.

Question A10.

(a) Find an element of order 6 in the symmetric group S_5.

(b) Show that there is no element of order 7 in S_5.

Question B1. Let V be a finite-dimensional complex inner product space of dimension at least 2, and $\mathbf{U}(V)$ the set of all isometries $f\colon V \to V$.

(a) Show that $\mathbf{U}(V)$ is a group under the operation of composition of linear transformations.

(b) Given an (ordered) orthonormal basis $\mathcal{B}$ of V , let $T(\mathcal{B})$ be the subset of $\mathcal{U}(V)$ consisting of those isometries f such that $[f]_{\mathcal{B}}$ is diagonal. Show that $T(\mathcal{B})$ is a subgroup, and is an abelian group.

(c) Show that for every $f \in \mathcal{U}(V)$ there exists an orthonormal basis $\mathcal{B}$ such that $f \in T(\mathcal{B})$.

(d) Show that if $\mathcal{B}$ and $\mathcal{B}'$ are orthonormal bases, then there exists $f \in \mathcal{U}(V)$ such that $fT(\mathcal{B})f^{-1} = T(\mathcal{B}')$.

(e) Is $T(\mathcal{B})$ a normal subgroup of $\mathbf{U}(V)$? Justify your answer.

Question B2. Let G be the symmetry group of a cube, and $\mathcal{F}$ the set of faces of the cube. You may use that G acts transitively on $\mathcal{F}$.

(a) Show (by exhibiting an isomorphism) that the stabiliser of a face $F_0 \in \mathcal{F}$ is isomorphic to the symmetry group of a square D_4.

(b) Use the orbit-stabilizer relation to compute the number of elements in G.

(c) Describe (either by drawing a picture or in words) an element of order 3 in G.

Question B3. Let G be a group.

(a) Define the notion of a conjugacy class in G.

(b) Show that every element of G is contained in precisely one conjugacy class.

(c) Suppose that G has exactly two conjugacy classes. Prove that G is a cyclic group with two elements.

(d) Give an example of a non-abelian group with exactly three conjugacy classes, and write down those conjugacy classes.

Question B4. Consider the complex vector space of 2×2 square matrices $V = M_2(\mathbb{C})$. Given $A \in M_2(\mathbb{C})$, we have a linear transformation $m_A \colon V \to V$ defined by $m_A(B) = AB$.

(a) Show: A and m_A have the same minimal polynomial.

(b) Show that if A is diagonalisable, then so is m_A.

(c) Suppose now that A is in Jordan normal form, that is, $A = J(a, 2)$, or $A = J(a, 1) \oplus J(b, 1)$. Find the Jordan normal form of m_A in each case.