Problem sheet 7

Groups, homomorphisms and examples

Vocabulary

(1) Define a group and give some illustrative examples.

(2) Define abelian group and give some illustrative examples.

(3) Define permutations and give some illustrative examples.

(4) Define a symmetric group and give some illustrative examples.

(5) Define a cyclic group and give some illustrative examples.

(6) Define cyclic subgroup generated by g and give some illustrative examples.

(7) Define $GL_n(\mathbb{R})$ and give some illustrative examples.

(8) Define $O_n(\mathbb{R})$ and give some illustrative examples.

(9) Define $U_n(\mathbb{C})$ and give some illustrative examples.

(10) Define $SL_n(\mathbb{R})$ and give some illustrative examples.

(11) Define $SO_n(\mathbb{R})$ and give some illustrative examples.

(12) Define $SU_n(\mathbb{C})$ and give some illustrative examples.

(13) Define subgroup and give some illustrative examples.

(14) Define subgroup generated by $g_1, \ldots, g_k$ and give some illustrative examples.

(15) Define order of a group and order of an element of a group and give some illustrative examples.

(16) Define homomorphism and isomorphism and give some illustrative examples.

(17) Define product of groups and give some illustrative examples.

(18) Define kernel and image of a group homomorphism and give some illustrative examples.

Results

(1) Let G be a group. Show that the identity of G is unique.

(2) Let G be a group and let $g, h \in G$. Show that $(gh)^{-1} = h^{-1}g^{-1}$.

(3) Let G be a group and let $g, x, y \in G$. Show that if $gx = gy$ then $x = y$.

(4) Let G be a group and let $g, h \in G$. Show that there exist unique $x, y \in G$ such that $gx = h$ and $yg = h$.

(5) Let G be a subgroup. Show that H is a subgroup of G if and only if H is a subset of G such that

$$\text{if } h_1, h_2 \in H \qquad \text{then} \qquad h_1h_2 \in H \text{ and } h_1^{-1} \in H.$$

(6) Let G be a subgroup. Show that H is a subgroup of G if and only if H is a subset of G such that

$$\text{if } h_1, h_2 \in H \qquad \text{then} \qquad h_1h_2^{-1} \in H.$$

(7) Show that every subgroup of a cyclic group is cyclic.

(8) Show that if G is a cyclic group then G is isomorphic to $\mathbb{Z}$ or there exists $n \in \mathbb{Z}_{>0}$ such that G is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

(9) Let $f: G \to H$ be a group homomorphism. Show that $f(1) = 1$.

(10) Let $f: G \to H$ be a group homomorphism and let $g \in G$. Show that $f(g^{-1}) = f(g)^{-1}$.

(11) Let $f: G \to H$ be a group homomorphism and let $g \in G$. Show that the order of g is equal to the order of $f(g)$.

(12) Let G and H be groups. Prove that $G \times H$ with operation defined by $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$ is a group.

Examples and computations

(1) Let n be a positive integer. Show that the set of all complex nth roots of unity $\{z \in \mathbb{C} \mid z^n = 1\}$ forms a group under multiplication.

(2) Let $U(n)$ be the set of $n \times n$ unitary matrices. Show that $U(n)$ is a group under matrix multiplication.

(3) Let G be a group and let $x, y, z\, w \in G$. Assume that $xyz^{-1}w = 1$. Solve for y.

(4) Compute the following products of permutations:

$$(123)(456) * (12)(34)(56) \qquad \text{and} \qquad (12) * (246) * (123654).$$

(5) Write out the multiplication table for the group S_3 of permutations of $\{1, 2, 3\}$ using cycle notation.

(6) Let G be a group and let $x, y, z \in G$. Assume that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that yxz=1?

(7) Assume that G is a group such that if $g \in G$ then $g^2 = 1$. Show that G is abelian.

(8) Show that $\mathbb{Z}$ with the operation of addition is a group.

(9) Show that $\mathbb{Q}$ with the operation of addition is a group.

(10) Show that $\mathbb{R}$ with the operation of addition is a group.

(11) Show that $\mathbb{C}$ with the operation of addition is a group.

(12) Show that $\mathbb{Z}$ with the operation of multiplication is not a group.

(13) Show that $\mathbb{Q}$ with the operation of multiplication is not a group.

(14) Show that $\mathbb{R}$ with the operation of multiplication is not a group.

(15) Show that $\mathbb{C}$ with the operation of multiplication is not a group.

(16) Show that $M_n(\mathbb{R})$ with the operation of addition is a group.

(17) Show that $GL_n(\mathbb{R})$ is a group.

(18) Show that $O_n(\mathbb{R})$ is a group.

(19) Show that $SL_n(\mathbb{R})$ is a group.

(20) Show that $SO_n(\mathbb{R})$ is a group.

(21) Describe the elements of $GL_1(\mathbb{R})$ and $GL_2(\mathbb{R})$.

(22) Describe the elements of $GL_1(\mathbb{Z})$ and $GL_2(\mathbb{Z})$.

(23) Describe the elements of $SL_1(\mathbb{R})$ and $SL_2(\mathbb{R})$.

(24) Describe the elements of $SL_1(\mathbb{Z})$ and $SL_2(\mathbb{Z})$.

(25) Describe the elements of $O_1(\mathbb{R})$ and $O_2(\mathbb{R})$.

(26) Describe the elements of $O_1(\mathbb{Z})$ and $O_2(\mathbb{Z})$.

(27) Describe the elements of $SO_1(\mathbb{R})$ and $SO_2(\mathbb{R})$.

(28) Describe the elements of $SO_1(\mathbb{Z})$ and $SO_2(\mathbb{Z})$.

(29) Describe the elements of $U_1(\mathbb{C})$, $SU_1(\mathbb{C})$, $U_2(\mathbb{C})$ and $SU_2(\mathbb{C})$.

(30) Describe the elements of $O_n(\mathbb{Z})$.

(31) Describe the elements of $SO_n(\mathbb{Z})$.

(32) Find the multiplication tables of all groups of order 2.

(33) Find the multiplication tables of all groups of order 3.

(34) Find the multiplication tables of all groups of order 4.

(35) Find the multiplication tables of all groups of order 5.

(36) Write the permutation $1 \to 2$, $2 \to 3$, $3 \to 1$, in diagram notation, in two line notation, in cycle notation, and in matrix notation.

(37) Write the permutation $1 \to 3$, $2 \to 2$, $3 \to 1$, in diagram notation, in two line notation, in cycle notation, and in matrix notation.

(38) Write the permutation $1 \to 3$, $2 \to 4$, $3 \to 5$, $4 \to 2$, $5 \to 1$, in diagram notation, in two line notation, in cycle notation, and in matrix notation.

(39) Calculate the products

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

(40) Write all elements of the symmetric group S_3 in diagram notation, in two line notation, in cycle notation, and in matrix notation.

(41) Determine whether the set of positive real numbers with the operation of addition is a group.

(42) Determine whether the set of $n \times n$ matrices over the real numbers with the operation of multiplication is a group.

(43) Let G be a group and let $x, y \in G$. Show that there exist unique w and z in G such that $wx = y$ and $xz = y$. Is $w = z$?

(44) Show that the set $\{z \in \mathbb{C} \mid n \in \mathbb{Z}_{>0}, zn = 1\}$ forms a group under multiplication.

(45) Compute the following products of permutations:

$$(123)(456)*(134)(25)(6), \qquad (12345)*(1234567) \qquad \text{and} \qquad (123456)*(123)*(123)*(1).$$

(46) Let $X = \mathbb{R} - \{0, 1\}$. Show that the following functions from X to X with the operation of composition of functions form a group:

$$f = \frac{1}{1-x}, \qquad g = \frac{x-1}{x}, \qquad h = \frac{1}{x}, \qquad i = x, \qquad j = 1 - x, \qquad k = \frac{x}{x-1}.$$

(47) Show that $2\mathbb{Z}$ is a subgroup of the group $\mathbb{Z}$.

(48) Show that the set of negative integers is not a subgroup of the group $\mathbb{Z}$.

(49) Let G be a group and let $1 \in G$. Show that $\{1\}$ is a subgroup of G.

(50) Show that $\{(1), (123), (132)\}$ is a subgroup of S_3.

(51) Show that $\{(1), (12), (23), (13)\}$ is not a subgroup of S_3.

(52) Let G be a group and let $g \in G$. Show that $\{g^n \mid n \in \mathbb{Z}\}$ is a subgroup of G.

(53) Let $n \in \mathbb{Z}_{>0}$. Calculate the order of $\mathbb{Z}/n\mathbb{Z}$, Always justify your answers.

(54) Let $n \in \mathbb{Z}_{>0}$. Calculate the order of S_n. Always justify your answers.

(55) Calculate the orders of the elements of $\mathbb{Z}/12\mathbb{Z}$, Always justify your answers.

(56) Calculate the orders of the elements of S_4, Always justify your answers.

(57) Show that S_3 is nonabelian and noncyclic.

(58) Define the function $\exp\colon \mathbb{R} \to \mathbb{R}_{>0}$ and prove that it is a group isomorphism.

(59) Prove that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are non-isomorphic groups of order 4.

(60) Prove that $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ are isomorphic groups of order 6.

(61) Prove that $\mathbb{Z}/6\mathbb{Z}$ and S_3 are nonisomorphic groups of order 6.

(62) Prove that the groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}2\mathbb{Z} \times \mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are all nonisomorphic.

(63) Calculate the order of $O_n(\mathbb{Z})$.

(64) Calculate the order of $SO_n(\mathbb{Z})$.

(65) Find the order of the element $(123)(4567)(89)$ in S_{10}.

(66) Find the order of the element $(14)(23567)$ in S_7.

(67) Find the orders of the elements 6, 12, 11, and 14 in $\mathbb{Z}/20\mathbb{Z}$.

(68) Find the orders of the elements 2, 12 and 8 in $\mathbb{Z}/13\mathbb{Z}$.

(69) Let G be a group and let $g \in G$.. Show that the order of g is equal to the order of g^{-1}.

(70) Let G be a commutative group and let $g, h \in G$. Show that if g and h have finite order then gh has finite order.