Problem sheet 5

Jordan normal form

Vocabulary

(1) Define Jordan block matrix and Jordan normal form and give some illustrative examples.

Results

(1) Let V be a vector space over $\mathbb{C}$ and let $f\colon V \to V$ be a linear transformation. Let $a \in \mathbb{C}$ and suppose that f has a minimal polynomial of the form $(x-a)^m$. Show that there is a basis of V with respect to which the matrix of f is triangular and $m \leqslant \dim(V)$.

(2) Let V be a vector space over $\mathbb{C}$ and let $f\colon V \to V$ be a linear transformation. Show that there is a basis of V with respect to which the matrix of f is triangular.

(3) Let V be a vector space and let f be a linear transformation on V. Let $c(x)$ be the characteristic polynomial of f. Show that $c(f) = 0$.

(4) Let V be a vector space over $\mathbb{C}$ and let $f\colon V \to V$ be a linear transformation. Show that there exists a basis for V with respect to which the matrix of f is

$$\begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix}$$

where each A_i is a Jordan block matrix. Show that this expression is unique up to reordering of the diagonal blocks $A_1, \ldots, A_k$.

(5) Let V be a vector space over $\mathbb{C}$ and let $f\colon V \to V$ be a linear transformation. Suppose that a is an eigenvalue of f and let $(x-a)^m$ be the highest power of $x-a$ dividing the minimal polynomial and let $(x-a)^n$ be the highest power of $x-a$ dividing the characteristic polynomial. Show that m is the size of the largest Jordan block of f with eigenvalue a and n is the sum of the sizes of the Jordan blocks of f with eigenvalue a.

(6) Let V be a vector space over $\mathbb{C}$ and let $f\colon V \to V$ be a linear transformation. Show that f can be represented by a diagonal matrix (with respect to a suitable basis) if and only if the minimal polynomial of f has no repeated roots. Show that if $\dim(V) = n$ and f has n distinct eigenvalues then f can be represented by a diagonal matrix.

Examples and computations

(1) In the vector space $\mathcal{P}_2(\mathbb{R})$ of polynomials with real coefficients and degree at most 2, decide whether the following set of three vectors is linearly independent, giving reasons for your answer: $\{1+2x, 1-x, 1+x+x^2\}$.

(2) If a linear transformation f on a finite dimensional vector space V satisfies $f^2 = f$, describe the possibilities for the Jordan normal form of f.

(3) Give an example of a 4× matrix over the real numbers which is not diagonalisable (that is, is not similar to a diagonal matrix). Give reasons for your answer.

(4) Given that the eigenvalues of the following matrix are 1, 1 and 2, calculate its Jordan Normal Form:

$$\begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & -1 & 4 \end{pmatrix}.$$

(5) Decide whether the following matrices span the space $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices with real entries:

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Always give reasons for your answers.

(6) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation represented by the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Describe a 2-dimensional f-invariant subspace of $\mathbb{R}^3$.

(7) Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}.$$

(8) Let $f\colon \mathbb{C}^3 \to \mathbb{C}^1$ denote the linear transformation given by

$$f(a_1, a_2, a_3) = (a_1 + a_2 + ia_3).$$

Find a basis for the kernel of f.

(9) Let A be a 6×6 matrix with complex entries. Suppose that the characteristic polynomial of A is known to be $x(x-1)^2(x-2)^3$. Given this information, what are the possibilities for the Jordan canonical form of A? What further computations could be used to establish which was the correct choice for the canonical form? Explain clearly how the outcome of your computations would enable you to determine the Jordan canonical form.

(10) Let A be a square matrix, with complex entries, of finite order. That is, $A^n = I$ for some natural number n, where I represents the identity matrix of appropriate size. Show that the minimal polynomial of A has no repeated roots. Deduce that A is diagonalisable; that is, A is similar to a diagonal matrix.

(11) Decide whether the following matrices span the space $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices with real entries:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Always give reasons for your answers.

(12) Decide whether the matrix

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$$

is diagonalisable (that is, is it similar to a diagonal matrix?) Always give proofs of your answers.

(13) A matrix is known to have minimal polynomial $(x-1)^2(x-2)(x-3)$ and characteristic polynomial $(x-1)^2(x-2)^2(x-3)$. Write down its Jordan normal form.

(14) Let f be a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$. If f^2 is the zero transformation, show that there is only one possible Jordan Normal form for f. If $g\colon \mathbb{R}^4 \to \mathbb{R}^4$ is non-zero and g^2 is zero, show that there are two possible Jordan Normal Forms for g. (For the purposes of this questions, we do not regard two Jordan Normal forms as being different if one can be obtained from the other by reordering the Jordan blocks.)

(15) Decide whether the following polynomials for a basis of the space $\mathcal{P}_2(\mathbb{R})$ of polynomials of degree at most 2 with real coefficients:

$$1-x+x^2, \qquad 1+x+x^2, \qquad 1+x-x^2.$$

Always give reasons for your answers.

(16) Find the minimal polynomial for the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Is the matrix diagonalisable? Always give proofs of your answers.

(17) A complex matrix has minimal polyomial $x^2(x-1)^2$ and characteristic polynomial $x^2(x-1)^4$. Find all the possibilities for the Jordan form of the matrix.

(18) $V = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$ is a vector space over the rational numbers $\mathbb{Q}$, using the usual operations of addition and multiplication for real numbers. Let $f\colon V \to V$ be multiplication by $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$ and a and b are not both zero.

(a) Show that f is a linear transformation.

(b) Find the matrix of f with respect to the basis $\{1, \sqrt{2}\}$ for V.

(c) Find the nullspace of f and find the rank of f.

(d) Is f surjective? Always prove your answers.

(19) Decide whether the set of polynomials

$$\{x^2+1, \quad x-1, \quad x^3+x, \quad x^4+2x^2\}$$

in the real vector space $P_4(\mathbb{R})$ of all polynomials of degree at most 4 with real coefficients is linearly independent or not. Does this collection of polynomials form a basis of $P_4(\mathbb{R})$?

(20) Let V be the complex vector space $\mathbb{C}^3$ and let $f\colon V \to V$ be the linear transformation given by

$$f(z_1, z_2, z_3) = (z_1, 5z_1 - 3z_2, -z_1 + iz_2 - (3-i)z_3).$$

Find the matrix of f relative to the basis $B = \{(i, 2, 0), (0, -1, 0), (0, 0, 3)\}$.

(21) A 6×6 complex matrix A has characteristic polynomial equal to $p(x) = (x+2)^3(x-2)^2(x-i)$. Determine all the possible Jordan normal forms for the matrix A (up to reordering of the Jordan blocks).

(22) Suppose that a linear transformation f from a real vector space V of dimension 8 to itself has minimum polynomial$x(x+1)^2(x-2)^3$.

(a) Describe the possibilities for the characteristic polynomial of f.

(b) List all the possible Jordan canonical forms of f.

(c) Explain why the linear transformation f cannot be invertible.

(23) Let V be the real vector space of all upper triangular 2×2 matrices with the usual operations of matrix addition and scalar multiplication:

$$V = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

(a) Write down a basis of V.

(b) Do the following matrices span V? Always prove your answers.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

(24) Let $V = \mathcal{P}_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 with real coefficients. We define a linear transformation $T\colon V \to V$ by $T(p(x)) = p(x+1) - p(x)$ for all polynomials $p(x)$ in V. Find the matrix of T with respect to the basis of V given by $B = \{1, x, x(x-1), x(x-1)(x-2)\}$.

(25) Find the minimal polynomial of the matrix

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

Is the matrix diagonalizable? Always prove your answers.

(26) A 5×5 complex matrix A has minimal polynomial $(x-2)^2(x-i)$. Determine all possible Jordan normal forms of A (up to rearranging the Jordan blocks).

(27) Let V be a complex vector space with an (ordered) basis $B = \{v_1, \ldots, v_k\}$, and let $T\colon V \to V$ be a linear transformation. Assume that the the matrix of T with respect to the basis B is a Jordan block

$$\begin{pmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \alpha \end{pmatrix}$$

(i) Write down the images $T(v_1), T(v_2), \dots, T(v_k)$.

(ii) Hence (or otherwise) write down the matrix of T with respect to the reversed basis $B' = \{v_k, \dots, v_2, v_1\}$.

(iii) Deduce that J is similar to its transpose J^t, i.e. there exists an invertible matrix P such that $P^{-1}JP = J^t$.

(28) Use the Jordan normal form theorem to prove that every $n \times n$ complex matrix A is similar to its transpose A^t.

(29) Let $V = \mathcal{P}_3(\mathbb{R})$ be the complex vector space of all polynomials of degree at most 3, with complex coefficients. Consider the polynomials

$$i, \quad x - i, \quad x^3 - 2ix + 3, \quad x^3 + 5.$$

(a) Are these polynomials linearly independent?

(b) Do they form a basis of V?

Always justify your answers.

(30) Consider the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 4 & 3 \\ 0 & -3 & -2 \end{pmatrix}.$$

(a) Find the minimal polynomial of A.

(b) Find the Jordan normal form for A.

(c) Is the matrix A diagonalisable?

(31) A complex matrix A has characteristic polynomial $(x+1)^2(x-7i)^4$ and minimal polynomial $(x+1)(x-7i)^2$.

(a) Determine all possible Jordan normal forms of A (up to rearranging the Jordan blocks).

(b) What additional information would allow you to determine the correct Jordan normal form?

(32) Let A be an $n \times n$ complex matrix, and let k be a positive integer.

(a) Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

(b) Conversely, use the Jordan normal form theorem to show that if μ is an eigenvalue of A^k then $\mu = \lambda^k$ for some eigenvalue λ of A.

(33) Let $V = \mathcal{P}_2(\mathbb{R})$ denote the vector space of polynomials in x of degree $\leqslant 2$ with real coefficients. Let $T\colon V \to V$ be the linear transformation defined by

$$T(p(x)) = p(x+1) + 3p'(x),$$

where p' is the derivative of p with respect to x.

(a) Find the matrix of T with respect to the basis $\{1, x, x^2\}$.

(b) Is T diagonalizable?

Always justify your answers.

(34) Consider the matrix
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 3 \end{pmatrix}.$$

(a) Find the minimal polynomial of A.

(b) Find the Jordan normal form of A.

(35) Let A be a 6×6 complex matrix with minimal polynomial
$$m(x) = (x+1)^2(x-1).$$

(a) Describe the possible characteristic polynomials for A.

(b) List the possible Jordan normal forms for A (up to reordering the Jordan blocks).

(c) Explain why A is invertible and write A^{-1} as a polynomial in A.

(36) Find the Jordan normal form of the matrix
$$A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

(37) Find the Jordan normal form of the matrix
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(38) Find the Jordan normal form of the matrix
$$A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(39) A complex matrix has characteristic polynomial $(x+2)^2(x-5)^3$. Describe all possibilities for the Jordan normal form (up to reordering of the Jordan blocks).

(40) Let A be a 5×5 complex matrix. Find the possible Jordan normal forms of A if A has characteristic polynomial $(x-\alpha)^5$ and $\operatorname{rank}(A - \alpha I) = 2$.

(41) Let A be a 5×5 complex matrix. Find the possible Jordan normal forms of A if A has characteristic polynomial $(x-2)^2(x-5)^3$, the $\lambda = 1$ eigenspace has dimension 1 and the $\lambda = 5$ eigenspace has dimension 2.

(42) Let $f\colon V \to V$ be a linear transformation on an n-dimensional complex vector space satisfying $f^4 = I =$ identity. Show that f is diagonalisable. What can you say about the eigenvalues of f and the Jordan normal form of f.

(43) Show that every 2×2 matrix A satisfies $A^2 - (\text{tr } A) + \det(A)I = 0$, where tr A is the trace of A.

(44) If A, B are 2×2 matrices with determinant 1, show that $A+A^{-1} = \operatorname{tr}(A)I$ and $B+B^{-1} = \operatorname{tr}(B)I$. Deduce that $\operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) = \operatorname{tr}(A)\operatorname{tr}(B)$.

(45) For any $n \times n$ complex matrix A define

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

If A and B are commuting matrices show that $e^{A+B} = e^A e^B$.

(46) Show that if J is a $k \times k$ Jordan block with α on the diagonal then $J = \alpha I + N$ with $N^k = 0$.

(47) Let J be a $k \times k$ Jordan block with α on the diagonal and $J = \alpha I + N$ with $N^k = 0$. Let t be a complex number. Show that $e^{tJ} = e^{\alpha t} e^{tN}$ and use this to calculate the matrices e^{tJ} for $k = 1, 2, 3, 4$.

(48) Let A be a $n \times n$ complex matrix. Show that $\det(e^A) = e^{\operatorname{tr}\ A}$.

(49) Show that the series defining e^A converges for all $n \times n$ complex matrices A.

(50) Does every $n \times n$ complex matrix have a complex square root?

(51) If f is a linear transformation on a finite dimensional vector space V satisfying $f^2 = f$, explain how to find a diagonal matrix representing f.

(52) Suppose that linear transformations f and g on a vector space V commute, i.e. $fg = gf$. Show that an eigenspace of f will be g-invariant. If the field $\mathbb{F}$ is algebraically closed, deduce that f and g have a common eigenvector.

(53) Find the Jordan normal form of the matrix $\begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix}$.

(54) Find the Jordan normal form of the matrix $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$.

(55) Find the Jordan normal form of the matrix $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$.

(56) Find the possible Jordan forms if the minimal polynomial is $x^2(x+1)^2$ and the characteristic polynomial is $x^2(x+1)^4$.

(57) Find the possible Jordan forms if the minimal polynomial is $(x-3)^2$ and the characteristic polynomial is $(x-3)^5$.

(58) Find the possible Jordan forms if the minimal polynomial is x^3 and the characteristic polynomial is x^7.

(59) Find the possible Jordan forms if the minimal polynomial is $(x-1)^2(x+1)^2$ and the characteristic polynomial is $(x-1)^4(x+1)^4$.

(60) Determine if the matrices $\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ are similar.

(61) Determine if the matrices $\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix}$ are similar.

(62) Determine if the matrices $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ are similar.

(63) Given a 4×4 matrix over A over $\mathbb{C}$ and given the minimal and characteristic polynomials of A, describe the possibilities for the Jordan Normal form of A.

(64) Show that any matrix in Jordan normal form is a sum $J = D + N$, where D is diagonal and N is nilpotent (i.e. $N^k = 0$ for some k). Deduce that any linear transformation f of a finite dimensional complex vector space can be written in the form $f = d + n$, where d is diagonalisable and n is nilpotent.

(65) Show that any matrix in Jordan normal form is a sum $J = D + N$, where D is diagonal, N is nilpotent, $JN = NJ$ and $JD = DJ$. Deduce that any linear transformation f of a finite dimensional complex vector space can be written in the form $f = d + n$, where d is diagonalisable, n is nilpotent, $fd = df$ and $fn = nf$.

(66) Show that the Jordan normal form of a complex matrix A is completely determined by the dimensions of the nullspaces of $(A - \lambda I)^i$ for the various eigenvalues λ of A and $i = 1, 2, 3, \ldots$.

(67) Let f be a reflection in $\mathbb{R}^3$. Show that the characteristic polynomial of f is $(x-1)^2(x+1)$ and the minimal polynomial is $x^2 - 1$.

(68) Let f be a linear transformation with matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that the characteristic polynomial is $(x-2)^2(x-1)$ and the minimal polynomial is $(x-2)(x-1)$.

(69) Write down the matrix in Jordan normal form that has one block of size 3 and eigenvalue 2, one block of size 4 and eigenvalue 0, and one block of size 1 and eigenvalue 4.

(70) Find the minimal polynomial, the characteristic polynomial and the Jordan normal form for the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

(71) Let $a, b \in \mathbb{C}$ Find the possible Jordan normal forms of 2×2 matrices that have characteristic polynomial $(x-a)(x-b)$.

(72) Let $a, b, c \in \mathbb{C}$ Find the possible Jordan normal forms of 2×2 matrices that have characteristic polynomial $(x-a)(x-b)(x-c)$.

(73) Find the minimal polynomial, the characteristic polynomial and the Jordan normal form for the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}.$$