MAST20022 Group Theory and Linear Algebra

Assignment 3

Due: 4pm Wednesday October 21, 2020

1. Determine whether the matrix $A = \begin{pmatrix} 3 & 4i \\ 4i & 3 \end{pmatrix}$ is
   (i) Hermitian, (ii) unitary, (iii) normal, (iv) diagonalizable.
   Always justify your answers.

2. Let V be a complex finite dimensional inner product space and let $f\colon V \to V$ be a linear transformation satisfying $f^*f = ff^*$.
   (a) State the spectral theorem and deduce that there is an orthonormal basis of V consisting of eigenvectors of f.
   (b) Show that there is a linear transformation $g\colon V \to V$ so that $f = g^2$.
   (c) Show that if every eigenvalue of f has absolute value 1, then $f^* = f^{-1}$.
   (d) Give an example to show that the result in (a) can fail if V is a real inner product space. (Hint: Consider the case $V = \mathbb{R}^2$.)

3. **First Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A p**-Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that G has a p-Sylow subgroup by completing the following steps.
   (a) Let $\Lambda^{p^a}(G)$ be the set of subsets of G of cardinality p^a. Show that if $j \in \{1, \ldots, p^a\}$ and p^i divides $p^a b - j$ then p^i divides $p^a - j$. Conclude that
   $$\mathrm{Card}(\Lambda^{p^a}(G)) = \binom{p^a b}{p^a} \quad \text{is not divisible by } p.$$
   (b) Consider the action of G on $\Lambda^{p^a}(G)$ by left multiplication and use
   $$\mathrm{Card}(\Lambda^{p^a}(G)) = \sum_{\text{distinct orbits}} \mathrm{Card}(GS),$$
   to conclude that there exists $S \in \Lambda^{p_a}(G)$ such that the cardinality of the orbit of S is not divisible by p.
   (c) Let $P = \mathrm{Stab}_G(S)$ and show that $\mathrm{Card}(P) = p^a$.

4. **Second Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A p**-Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that all p-Sylow subgroups of G are conjugate by completing the following steps.
   (a) Let P and H be p-Sylow subgroups of G. Let H act on G/P by left multiplication. Use
   $$\mathrm{Card}(G/P) = \sum_{\text{distinct orbits}} \mathrm{Card}(HgP),$$
   to show that there is an orbit HgP with $\mathrm{Card}(HgP) = 1$.

(b) Show that $H \subseteq gPg^{-1}$ and conclude that $H = gPg^{-1}$.

5. **Third Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A p-**Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that the number of p-Sylow subgroups of G is 1 mod p by completing the following steps.
   (a) Let P be a p-Sylow subgroup of G. Let P act on the set $\mathcal{S}$ of p-Sylow subgroups of G by conjugation. Show that if $P*Q$ is an orbit under this action then $\mathrm{Card}(P*Q) = 1$ or p divides $\mathrm{Card}(P*Q)$.
   (b) Assume $\mathrm{Card}(P*Q) = 1$ and let $N(Q)$ be the normalizer of Q. Show that both P and Q are both p-Sylow subgroups of $N(Q)$.
   (c) Assume $\mathrm{Card}(P*Q) = 1$. Use the second Sylow theorem and part (b) to show that $P = Q$.
   (d) Use part (a) and (c) and
   $$\mathrm{Card}(\mathcal{S}) = \sum_{\text{distinct orbits}} \mathrm{Card}(P*Q)$$
   to conclude that $\mathrm{Card}(\mathcal{S}) = 1 \bmod p$.

6. **Fourth Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A p-**Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that the number of p-Sylow subgroups divides $\mathrm{Card}(G)$ by completing the following steps.
   (a) Let G act on the set $\mathcal{P}$ of p-Sylow subgroups of G by conjugation. Use the second Sylow theorem to conclude that there is only one orbit under this action.
   (b) Conclude, from (a), that the number of p-Sylow subgroups divides $\mathrm{Card}(G)$.