CHAPTER 1

NUMBER SYSTEMS

An **ordered commutative ring** is a set A with a relation $\leqslant$ on A and functions

$$\begin{array}{ccc} A\times A & \longrightarrow & A \\ (a,b) & \longmapsto & a+b \end{array} \quad \text{and} \quad \begin{array}{ccc} A\times A & \longrightarrow & A \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Fa) If $a, b, c \in A$ then $(a+b)+c = a+(b+c)$,
(Fb) If $a, b \in A$ then $a+b = b+a$,
(Fc) There exists $0 \in A$ such that

$$\text{if } a \in A \quad \text{then} \quad 0+a = a \text{ and } a+0 = a,$$

(Fd) If $a \in A$ then there exists $-a \in A$ such that $a+(-a) = 0$ and $(-a)+a = 0$,
(Fe) If $a, b, c \in A$ then $(ab)c = a(bc)$,
(Ff) If $a, b, c \in A$ then

$$(a+b)c = ac+bc \qquad \text{and} \qquad c(a+b) = ca+cb,$$

(Fg) There exists $1 \in A$ such that

$$\text{if } a \in A \quad \text{then} \quad 1\cdot a = a \text{ and } a\cdot 1 = a.$$

(Fi) If $a, b \in A$ then $ab = ba$.

(Pa) If $x \in A$ then $x \leqslant x$,
(Pb) If $x, y, z \in A$ and $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$, and
(Pc) If $x, y \in A$ and $x \leqslant y$ and $y \leqslant x$ then $x = y$.
(Pd) If $x, y \in A$ then $x \leqslant y$ or $y \leqslant x$.

(OFa) If $a, b, c \in A$ and $a \leqslant b$ then $a+c \leqslant b+c$,
(OFb) If $a, b \in A$ and $a \geqslant 0$ and $b \geqslant 0$ then $ab \geqslant 0$.

An ordered commutative ring A satisfies the **cancellation property** if it satisfies:

(CP) if $a, b, c \in A$ and $c \neq 0$ and $ac = bc$ then $a = b$.

An **ordered field** is an ordered commutative ring $\mathbb{F}$ such that

(Fh) If $a \in \mathbb{F}$ and $a \neq 0$ then there exists $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$ and $a^{-1}a = 1$,

1.1. The integers $\mathbb{Z}$

The **positive integers** is the set

$$\mathbb{Z}_{>0} = \{1, 1+1, 1+1+1, 1+1+1+1, \ldots\}$$

with operation given by concatenation so that, for example,

$$(1+1+1) + (1+1+1+1) = 1+1+1+1+1+1+1.$$

The positive integers are often written as

$$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}.$$

If $x, y \in \mathbb{Z}_{>0}$ write

$$x < y \quad \text{if there exists } n \in \mathbb{Z}_{>0} \text{ such that } x + n = y.$$

If $x, y \in \mathbb{Z}_{>0}$ then $x < y$ or $x > y$ or $x = y$.

The **nonnegative integers** is the set

$$\mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3, \ldots\}$$

with operation given by the addition in $\mathbb{Z}_{>0}$ and

(Z0) if $x \in \mathbb{Z}_{\geqslant 0}$ then $\quad 0 + x = x \quad$ and $\quad x + 0 = x$.

The integers is the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

with operation given by the addition in $\mathbb{Z}_{\geqslant 0}$ and

(Za) If $x, y \in \mathbb{Z}_{>0}$ and $x < y$ then $(-x) + y = n$ and $y + (-x) = n$, where $n \in \mathbb{Z}_{>0}$ is such that $x + n = y$.
(Zb) If $x, y \in \mathbb{Z}_{>0}$ and $x > y$ then $(-x) + y = m$ and $y + (-x) = m$, where $m \in \mathbb{Z}_{>0}$ is such that $y + m = x$.
(Zc) if $x \in \mathbb{Z}_{>0}$ then $(-x) + x = 0$, and $x + (-x) = 0$,
(Zd) If $x, y \in \mathbb{Z}_{>0}$ then $(-x) + (-y) = -(x + y)$.
(Ze) If $x \in \mathbb{Z}_{>0}$ then $0 + (-x) = (-x) + 0 = -x$.

Proposition 1.1.1. — *Let $k \in \mathbb{Z}$. Then there exists a unique function $m_k \colon \mathbb{Z} \to \mathbb{Z}$ such that*

(a) *If $x, y \in \mathbb{Z}$ then $m_k(x + y) = m_k(x) + m_k(y)$, and*
(b) *$m_k(1) = k$.*

HW: Show that if $k \in \mathbb{Z}$ and $z \neq 0$ then m_k is injective.

The **multiplication** on $\mathbb{Z}$ is

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (k, l) & \mapsto & kl = m_k(l). \end{array}$$

Define a relation $\leqslant$ on $\mathbb{Z}$ by

$$x \leqslant y \qquad \text{if there exists } n \in \mathbb{Z}_{\geqslant 0} \text{ such that } x + n = y.$$

Theorem 1.1.2. — *The set $\mathbb{Z}$ with the operations of addition and multiplication and the total order $\leqslant$ is an ordered commutative ring which satisfies the cancellation property.*