F.3. Exercises: Vector spaces

Exercise F.3.1. — Let $\mathbb{F}$ be a field.

(a) Show that the intersection of two subspaces of an $\mathbb{F}$-vector space V is a subspace of V.
(b) Give an example to show that the union of two subspaces of an $\mathbb{F}$-vector space V is not necessarily a subspace of V.
(c) Let W and U be subspaces of an $\mathbb{F}$-vector space V. Show that $W + U = \{w + u \mid w \in W \text{ and } u \in U\}$ is a subspace of V.
(d) Let W and U be subspaces of an $\mathbb{F}$-vector space V. Show that $V \simeq W \oplus U$ if and only if $W \cap U = (0)$ and $V = W + U$.

Exercise F.3.2. — Let V be an $\mathbb{F}$-vector space and let S be a subset of V. Let $\mathcal{W}$ be the set of subspaces W of V such that $S \subseteq W$. Define

$$W_S = \bigcap_{W \in \mathcal{W}} W.$$

(a) Show that W_S is a subspace of V.
(b) Show that $S \subseteq W_S$ since $S \subseteq W$ for every $W \in \mathcal{W}$.
(c) Show that if W is a subspace of V and $S \subseteq W$ then $W \supseteq W_S$.

Conclude that $W_S = \text{span}_{\mathbb{F}}(S)$. So $\text{span}_{\mathbb{F}}(S)$ is the smallest subspace of V containing S.

Exercise F.3.3. — Let V be an $\mathbb{F}$-vector space and let S be a subset of V. A **linear combination** of elements of S is an element of V of the form

$$\sum_{s \in S} c_s s$$

where $c_s \in \mathbb{F}$ and all but a finite number of the values c_s are equal to 0 (the set S may be infinite but we do not want to take infinite sums).

(a) Let W be a subspace of V. Show that a linear combination of elements of W is an element of W.
(b) Give an example of a vector space V, a subset $S \subseteq W$, and a linear combination v of elements of S such that $v \notin S$.
(c) Let S be a subset of V and let L_S be the set of all linear combinations of elements of S.
  (ca) Show that L_S is a subspace of V.
  (cb) Show that $L_S = \text{span}_{\mathbb{F}}(S)$.

Exercise F.3.4. — Let $\mathbb{F}$ be a field. A **column vector** of length n is an $n \times 1$ array

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \qquad \text{of elements } c_i \in \mathbb{F}.$$

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Define an addition operation on column vectors by

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{pmatrix} = \begin{pmatrix} c_1 + c'_1 \\ c_2 + c'_2 \\ \vdots \\ c_n + c'_n \end{pmatrix}$$

Define an action of $\mathbb{F}$ on column vectors by

$$c \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{pmatrix}$$

The action of $\mathbb{F}$ is **scalar multiplication**.
Show that the set $\mathbb{F}^n$ of column vectors of length n is an $\mathbb{F}$-vector space.

Exercise F.3.5. — Let $T\colon V \to U$ be a linear transformation.

(a) Let $W \subseteq V$ be a subspace of V and define

$$T(W) = \{T(w) \mid w \in W\}.$$

(aa) Show that $T(W) \subseteq \mathrm{im}T = T(V)$.
(ab) Show that $T(W)$ is a subspace of U.

$$\begin{array}{ccc} V & \xrightarrow{T} & U \\ \\ W & \longmapsto & T(W) \\ \cap| & & \cap| \\ V & \longmapsto & T(V) \quad = \mathrm{im}T. \end{array}$$

(b) Let Y be a subspace of U and define

$$T^{-1}(Y) = \{v \in V \mid T(v) \in Y\}.$$

(ba) Show that $T^{-1}(Y) \supseteq \ker T = T^{-1}\big((0)\big)$.
(bb) Show that $T^{-1}(Y)$ is a subspace of V.

$$\begin{array}{rccc} & V & \xrightarrow{T} & U \\ \\ & T^{-1}(Y) & \longmapsto & Y \\ & \cup| & & \cup| \\ \ker T = & T^{-1}\big((0)\big) & \longmapsto & (0). \end{array}$$

(c) (ca) Let W be a subspace of V and show that $W \subseteq T^{-1}\big(T(W)\big)$.
(cb) Give an example of a linear transformation $T\colon V \to U$ and a subspace W of V such that $W \neq T^{-1}\big(T(W)\big)$.
(cc) Show that if W is a subspace of V that contains $\ker T$ then $W = T^{-1}\big(T(W)\big)$.

(d) (da) Let Y be a subspace of U and show that $T\big(T^{-1}(Y)\big) \subseteq Y$.
(db) Give an example of a linear transformation $T\colon V \to U$ and a subspace Y such that $T\big(T^{-1}(Y)\big) \neq Y$.
(dc) Show that if Y is a subspace of U and $Y \subseteq \mathrm{im}T$ then $Y = T\big(T^{-1}(Y)\big)$.

(e) Conclude from (c) and (d) that there is a one-to-one correspondence between subspaces of V that contain $\ker T$ and subspaces of U that are contained in $\mathrm{im}\, T$.

$$\{\text{subspaces of } V \text{ containing } \ker T\} \quad \longleftrightarrow \quad \{\text{subspaces of } U \text{ contained in } \mathrm{im}\, T\}$$

Exercise F.3.6. — Let $\mathbb{F}$ be a field.

(a) Let W be a subspace of an $\mathbb{F}$-vector space V. The **inclusion** is the function

$$\begin{array}{rccc} \iota\colon & W & \to & V \\ & w & \mapsto & w. \end{array}$$

Show that $\iota\colon W \to V$ is a well defined injective linear transformation.

(b) Let W be a subspace of a $\mathbb{F}$-vector space V. The **quotient map** is the function

$$\begin{array}{rccc} \pi\colon & V & \to & V/W \\ & v & \mapsto & v+W. \end{array}$$

Show that $\pi\colon V \to V/W$ is a well defined surjective linear transformation and that $\mathrm{im}\pi = V/W$ and $\ker \pi = W$.

(c) Let U be a subspace of V. Show that

(ca) $U/W = \{u + W \mid u \in U\}$ is a subspace of V/W.

(cb) $U/W = \pi(U)$ and if U contains W then $\pi^{-1}\big(\pi(U)\big) = U$.

(cc) Conclude that there is a one-to-one correspondence

$$\{\text{subspaces of } V \text{ containing } W\} \quad \longleftrightarrow \quad \{\text{subspaces of } V/W\}.$$

Exercise F.3.7. — Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let W be a subspace of V and let U be a subspace of V containing W. Then, by Ex. F.3.6(ca), U/W is a subspace of V/W.

Let $\frac{V/W}{U/W}$ be the quotient space and let

$$\pi_2\colon V/W \to \frac{V/W}{U/W} \qquad \text{be the quotient map.}$$

Let $\pi_1\colon V \to V/W$ be the quotient map so that

$$(\pi_1 \circ \pi_2)\colon V \xrightarrow{\pi_1} V/W \xrightarrow{\pi_2} \frac{V/W}{U/W}.$$

(a) Show that $\mathrm{im}(\pi_1 \circ \pi_2) = \dfrac{V/W}{U/W}$.

(b) Show that $\ker(\pi_1 \circ \pi_2) = U$.

(c) Using Theorem F.2.6(c), conclude that $V/U \simeq \dfrac{V/W}{U/W}$ as vector spaces.

Exercise F.3.8. — Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let W be a subspace of V and let U be any subspace of V. Let

$$\begin{array}{rccc} \pi\colon & U & \to & V/W \\ & u & \mapsto & u+W \end{array}$$

be the restriction of the quotient map $\pi\colon V \to V/W$ to U.

(a) Show that $\ker \pi = U \cap W$.

(b) Show that $\mathrm{im}\pi = \dfrac{U+W}{W} = \{u + W \mid u \in U\}$.

(c) Using Theorem F.2.6(c), conclude that $\dfrac{U}{U \cap W} \simeq \dfrac{U+W}{W}$.

Exercise F.3.9. — Let $\mathbb{F}$ be a field. Let W_1 be a subspace of an $\mathbb{F}$-vector space V_1 and let W_2 be a subspace of an $\mathbb{F}$-vector space V_2.

(a) Show that $W_1 \oplus W_2$ is a subspace of the $\mathbb{F}$-vector space $V_1 \oplus V_2$.

(b) Let $\pi_1 \colon V_1 \to V_1/W_1$ and $\pi_2 \colon V_2 \to V_2/W_2$ be the quotient maps. Define a map

$$\begin{array}{rccc} (\pi_1 \oplus \pi_2)\colon & V_1 \oplus V_2 & \to & V_1/W_1 \oplus V_2/W_2 \\ & (v_1, v_2) & \mapsto & (v_1 + W_1, v_2 + W_2). \end{array}$$

Show that $\pi_1 \oplus \pi_2$ is a well defined surjective linear transformation.

(c) Show that $\ker(\pi_1 \oplus \pi_2) = W_1 \oplus W_2$.

(d) Using Theorem F.2.6(c), conclude that

$$\frac{V_1 \oplus V_2}{W_1 \oplus W_2} \simeq \frac{V_1}{W_1} \oplus \frac{V_2}{W_2}.$$