R.3. Exercises: Rings

Exercise R.3.1. — Let R be a ring.

(a) Show that the intersection of two subrings of R is a subring of R.
(b) Give an example which shows that the union of two subrings of a ring is not necessarily a subring.

Let I and J be ideals of R.

(c) Show that $I \cap J$ is an ideal of R.
(d) Give an example to show that $I \cup J$ is not necessarily an ideal of R.
(e) Show that $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$ is an ideal of R.
(f) Give an example to show that the set $\{ij \mid i \in I \text{ and } j \in J\}$ is not necessarily an ideal of R.

Exercise R.3.2. — Let R be a ring and let S be a subset of R. Let $\mathcal{J}$ be the set of ideals I of R such that $S \subseteq I$. Define

$$I_S = \bigcap_{I \in \mathcal{J}} I.$$

(a) Show that I_S is an ideal of R.
(b) Show that $S \subseteq I_S$.
(c) Show that if I is an ideal of R and $S \subseteq I$ then $I_S \subseteq I$.

Conclude that $I_S = (S)$. So (S) is the "smallest" ideal of R containing S.

Exercise R.3.3. — Give an example of two rings R and S and a function $f\colon R \to S$ such that

(a) $f(r_1 + r_2) = f(r_1) + f(r_2)$, and
(b) $f(r_1 r_2) = f(r_1)f(r_2)$ for all $r_1, r_2 \in R$, but such that
(c) $f(1_R) \neq 1_S$.

This shows that conditions (a) and (b) in the definition of ring homomorphism do not imply condition (c). What is different between this case and the case in Theorem G.1.1(a) where $f(1_G) = 1_H$ for a function that satisfies (a)?

Exercise R.3.4. — Let $f\colon R \to S$ be a ring homomorphism.

(a) Let $I \subseteq R$ be an ideal of R and define

$$f(I) = \{f(i) \mid i \in I\}.$$

(aa) Show that $f(I) \subseteq \mathrm{im} f = f(R)$.
(ab) Show that if f is surjective then $f(I)$ is an ideal of S.
(ac) Give an example of a homomorphism $f\colon R \to S$ and an ideal I of R such that $f(I)$ is not an ideal of S.

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \\ I & \longmapsto & f(I) \\ \cap\!\mid & & \cap\!\mid \\ R & \longmapsto & f(R) \quad = \mathrm{im} f. \end{array}$$

(b) Let J be an ideal of S and define

$$f^{-1}(J) = \{r \in R \mid f(r) \in J\}.$$

(ba) Show that $f^{-1}(J) \supseteq \ker f = f^{-1}\big((0)\big)$.
(bb) Show that $f^{-1}(J)$ is an ideal of R.

$$\begin{array}{rccc} & R & \xrightarrow{f} & S \\ \\ & f^{-1}(J) & \longmapsto & J \\ & \bigcup\!\mid & & \bigcup\!\mid \\ \ker f = & f^{-1}\big((0)\big) & \longmapsto & (0). \end{array}$$

(c) (ca) Let I be an ideal of R and show that $I \subseteq f^{-1}\big(f(I)\big)$.
(cb) Give an example of a homomorphism $f\colon R \to S$ and an ideal I of R such that $I \neq f^{-1}\big(f(I)\big)$.
(cc) Show that if I is an ideal of R that contains $\ker f$ then $I = f^{-1}\big(f(I)\big)$.

(d) (da) Let J be an ideal of S and show that $f\big(f^{-1}(J)\big) \subseteq J$.
(db) Give an example of a homomorphism $f\colon R \to S$ and an ideal J of S such that $f\big(f^{-1}(J)\big) \neq J$.
(dc) Show that if J is an ideal of S and $J \subseteq \mathrm{im} f$ then $J = f\big(f^{-1}(J)\big)$.

(e) (ea) It follows from Ex. G.3.5 that there is a one-to-one correspondence between additive subgroups of R containing $\ker f$ and additive subgroups of S that are contained in $\mathrm{im} f$. Give an example to show that this correspondence does not necessarily take ideals to ideals.

$$\{\text{subgroups of } R \text{ containing } \ker f\} \quad \longleftrightarrow \quad \{\text{subgroups of } S \text{ contained in } \mathrm{im}\, f\}$$

(ea) Show that if $f\colon R \to S$ is surjective then this correspondence takes ideals to ideals.

Exercise R.3.5. — Let I be an ideal of a ring R.

(a) The **inclusion** is the map $\iota\colon I \to R$ given by

$$\begin{array}{rccc} \iota\colon & I & \to & R \\ & i & \mapsto & i \end{array}$$

Show that $\iota\colon I \to R$ is a well defined injective homomorphism.

(b) The **quotient map** is the map $\pi\colon R \to R/I$ given by

$$\begin{array}{rccc} \pi\colon & R & \to & R/I \\ & r & \mapsto & r+I. \end{array}$$

Show that $\pi\colon R \to R/I$ is a well defined surjective homomorphism and that $\mathrm{im}\,\pi = R/I$ and $\ker \pi = I$.

(c) Let J be an ideal of R. Show, using Ex. R.3.4, that
(ca) $J/I = \{j + I \mid j \in J\}$ is an ideal of R/I.
(cb) $J/I = \pi(J)$ and if J contains I then $\pi^{-1}\big(\pi(J)\big) = J$.
(cc) Conclude that there is a one-to-one correspondence between ideals of R containing I and ideals of R/I.

$$\{\text{ideals of } R \text{ containing } I\} \quad \longleftrightarrow \quad \{\text{ideals of } R/I\}$$

Exercise R.3.6. — An **exact sequence**

$$\cdots \longrightarrow R_{i-1} \xrightarrow{f_{i-1}} R_i \xrightarrow{f_i} R_{i+i} \longrightarrow \cdots$$

is a sequence of rings and ring homomorphisms $f_i\colon R_i \to R_{i+1}$, such that

$$\ker f_i = \mathrm{im} f_{i-1}.$$

A **short exact sequence** is an exact sequence of the form

$$(0) \longrightarrow I \xrightarrow{g} R \xrightarrow{f} S \longrightarrow (0).$$

(a) Show that in the above short exact sequence g is always injective and f is always surjective.
(b) Let $f\colon R \to S$ be a homomorphism and show that the sequence

$$(0) \to \ker f \to R \to \mathrm{im} f \to (0) \qquad \text{is exact.}$$

(c) Let I be an ideal of a ring R. Let $\iota\colon I \to R$ be the canonical injection and let $\pi\colon R \to R/I$ be the canonical surjection. Show that

$$(0) \to I \xrightarrow{\iota} R \xrightarrow{\pi} R/I \to (0) \qquad \text{is a short exact sequence.}$$

Exercise R.3.7. — Let I be an ideal of R and let J be an ideal of R containing I. Then, by Ex. R.3.5(ca), J/I is an ideal of R/I.

Let $\frac{R/I}{J/I}$ be the quotient ring and let

$$\pi_2\colon R/I \to \frac{R/I}{J/I}$$

be the canonical surjection.

Let $\pi_1\colon R \to R/I$ be the canonical surjection so that

$$(\pi_1 \circ \pi_2)\colon R \xrightarrow{\pi_1} R/I \xrightarrow{\pi_2} \frac{R/I}{J/I}.$$

(a) Show that $\mathrm{im}(\pi_1 \circ \pi_2) = \dfrac{R/I}{J/I}$.
(b) Show that $\ker(\pi_1 \circ \pi_2) = J$.
(c) Using Theorem R.1.6, conclude that $R/J \simeq \dfrac{R/I}{J/I}$ as rings.

Exercise R.3.8. — Let I be an ideal of R and let S be any subring of R. Let

$$\begin{array}{rccc} \pi\colon & S & \to & R/I \\ & s & \mapsto & s+I \end{array}$$

be the restriction of the canonical surjection $\pi\colon R \to R/I$ to S.

(a) Show that $\ker \pi = S \cap I$.
(b) Show that $\mathrm{im}\, \pi = \dfrac{S+I}{I} = \{s + I \mid s \in S\}$.
(c) Using Theorem R.1.6, conclude that $\dfrac{S}{S \cap I} \simeq \dfrac{S+I}{I}$.

Exercise R.3.9. — Let I_1 and I_2 be ideals of rings R_1 and R_2 respectively. Define $I_1 \oplus I_2$ to be the subset of $R_1 \oplus R_2$ given by

$$I_1 \oplus I_2 = \{(i_1, i_2) \mid i_1 \in I_1 \text{ and } i_2 \in I_2\}.$$

(a) Show that $I_1 \oplus I_2$ is an ideal of the ring $R_1 \oplus R_2$.
(b) Let $\pi_1\colon R_1 \to R_1/I_1$ and $\pi_2\colon R_2 \to R_2/I_2$ be the canonical projections. Define a map

$$\begin{array}{rccc} (\pi_1 \oplus \pi_2)\colon & R_1 \oplus R_2 & \to & (R_1/I_1) \oplus R_2/I_2 \\ & (r_1, r_2) & \mapsto & (r_1 + I_1, r_2 + I_2). \end{array}$$

Show that $\pi_1 \oplus \pi_2$ is a well defined surjective homomorphism.

(c) Show that $\ker(\pi_1 \oplus \pi_2) = I_1 \oplus I_2$.

(d) Using Theorem R.1.6, conclude that

$$\frac{R_1 \oplus R_2}{I_1 \oplus I_2} \simeq \frac{R_1}{I_1} \oplus \frac{R_2}{I_2}.$$