G.3. Exercises: Groups

Exercise G.3.1. —

(a) Show that the intersection of two subgroups of a group G is a subgroup of G.
(b) Give an example which shows that the union of two subgroups of a group G is not necessarily a subgroup of G.

Exercise G.3.2. — Let G be a group and let S be a subset of G. Let $\mathcal{H}$ be the set of subgroups H of G such that $S \subseteq H$.
Define

$$H_S = \bigcap_{H\in\mathcal{H}} H.$$

(a) Show that H_S is a subgroup of G.
(b) Show that if $H \in \mathcal{H}$ then $S \subseteq H$ and $S \subseteq H_S$.
(c) Show that if H is a subgroup of G and $S \subseteq H$ then $H \supseteq H_S$.
(d) Conclude that $H_S = \langle S\rangle$.

So $\langle S\rangle$ is the "smallest" subgroup of G containing S.

Exercise G.3.3. — **Lagrange's Theorem.** Let G be a group and let H and K be subgroups of G with $K \subseteq H \subseteq G$. Show that

$$\mathrm{Card}(G/K) = \mathrm{Card}(G/H)\mathrm{Card}(H/K).$$

Show that Corollary G.1.4 is a special case of this theorem with $K = \{1\}$.

Exercise G.3.4. — Let G be a group and let H be a subgroup of G. A **double coset** of H in G is a set

$$HgH = \{hgh' \mid h, h' \in H\} \quad \text{where } g \in G.$$

Let $H\backslash G/H$ be the set of double cosets of H in G.

Show that the double cosets of H in G partition G.

Exercise G.3.5. — Let $f\colon G \to H$ be a group homomorphism.

(a) Let $M \subseteq G$ be a subgroup of G and define

$$f(M) = \{f(m) \mid m \in M\}.$$

(aa) Show that $f(M)$ is a subgroup of G.
(ab) Show that $f(M) \subseteq \mathrm{im} f = f(G)$.

$$\begin{array}{ccccc} G & \xrightarrow{f} & H & & \\ & & & & \\ M & \longmapsto & f(M) & & \\ \cap\!| & & \cap\!| & & \\ G & \longmapsto & f(G) & = \mathrm{im} f. & \end{array}$$

(ac) Show that if f is surjective and M is a normal subgroup of G then $f(M)$ is a normal subgroup of H.
(ad) Give an example of a homomorphism $f\colon G \to H$ and of a normal subgroup M of G such that $f(M)$ is not a normal subgroup of H.

(b) Let $N \subseteq H$ be a subgroup of H and define

$$f^{-1}(N) = \{g \in G \mid f(g) \in N\}.$$

(ba) Show that $f^{-1}(N)$ is a subgroup of G.
(bb) Show that $f^{-1}(N) \supseteq \ker f = f^{-1}(1)$.

$$\begin{array}{rccc} & G & \xrightarrow{f} & H \\ & & & \\ & f^{-1}(N) & \longmapsto & N \\ & \cup\!| & & \cup\!| \\ \ker f = & f^{-1}(1) & \longmapsto & (1). \end{array}$$

(bc) Show that if N is a normal subgroup of H then $f^{-1}(N)$ is a normal subgroup of G.

(c) (ca) Let M be a subgroup of G and show that $M \subseteq f^{-1}\big(f(M)\big)$.
(cb) Give an example of a homomorphism $f: G \to H$ and a subgroup M of G such that $M \neq f^{-1}\big(f(M)\big)$.
(cc) Show that if $M \subseteq G$ is a subgroup of G that contains $\ker f$ then $M = f^{-1}\big(f(M)\big)$.

(d) (da) Let N be a subgroup of H and show that $f\big(f^{-1}(N)\big) \subseteq N$.
(db) Give an example of a homomorphism $f: G \to H$ and a subgroup N of H such that $N \neq f\big(f^{-1}(N)\big)$.
(dc) Show that if $N \subseteq H$ is a subgroup of H and $N \subseteq \mathrm{im} f$ then $N = f\big(f^{-1}(N)\big)$.

(e) (ea) Conclude from (c) and (d) that there is a one-to-one correspondence

$$\{\text{subgroups of } G \text{ that contain } \ker f\} \longleftrightarrow \{\text{subgroups of } H \text{ contained in } \mathrm{im} f\}.$$

(eb) Give an example to show that this correspondence does not necessarily take normal subgroups to normal subgroups.
(ec) Show that if f is surjective then this correspondence takes normal subgroups of G to normal subgroups of H.

Exercise G.3.6. —

(a) Let H be a subgroup of a group G. The **inclusion** is the function

$$\begin{array}{rccc} \iota: & H & \to & G \\ & h & \mapsto & h. \end{array}$$

Show that $\iota: H \to G$ is a well defined injective homomorphism.

(b) Let N be a normal subgroup of a group G. The **quotient map** is the function

$$\begin{array}{rccc} \pi: & G & \to & G/N \\ & g & \mapsto & gN. \end{array}$$

Show that $\pi: G \to G/N$ is a well defined surjective homomorphism and that $\mathrm{im}\,\pi = G/N$ and $\ker \pi = N$.

(c) Let M be a subgroup of G. Show, using Ex. G.3.5, that
(ca) $M/N = \{mN | m \in M\}$ is a subgroup of G/N.
(cb) M/N is a normal subgroup of G/N if M is a normal subgroup of G.
(cc) $M/N = \pi(M)$ and if M contains N then $\pi^{-1}\big(\pi(M)\big) = M$.
(cd) Conclude that there is a one-to-one correspondence

$$\{\text{subgroups of } G \text{ containing } N\} \longleftrightarrow \{\text{subgroups of } G/N\}.$$

(ce) Show that *this correspondence takes normal subgroups to normal subgroups.*

Exercise G.3.7. — An **exact sequence**

$$\cdots \longrightarrow G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+i} \longrightarrow \cdots$$

is a sequence of group homomorphisms $f_i \colon G_i \to G_{i+1}$ such that

$$\ker f_i = \mathrm{im} f_{i-1}.$$

A **short exact sequence** is an exact sequence of the form

$$(1) \to K \xrightarrow{g} G \xrightarrow{f} H \to (1).$$

(a) Show that if $(1) \to K \xrightarrow{g} G \xrightarrow{f} H \to (1)$. is a short exact sequence then g is injective and f is surjective.
(b) Let $f \colon G \to H$ be a homomorphism and let $\iota\colon \ker f \to G$ be the canonical injection. Show that the sequence

$$(1) \to \ker f \xrightarrow{\iota} G \xrightarrow{f} \mathrm{im} f \to (1) \qquad \text{is exact.}$$

(c) Let K be a normal subgoup of a group G. Let $\iota\colon K \to G$ be the canonical injection and let $\pi\colon G \to G/K$ be the canonical surjection. Show that

$$(1) \to K \xrightarrow{\iota} G \xrightarrow{\pi} G/K \to (1) \qquad \text{is a short exact sequence.}$$

Exercise G.3.8. — Let N be a normal subgroup of a group G. Let K be a normal subgroup of G containing N. Then, by Ex. G.3.6(cb), $K/N = \{kN \mid k \in K\}$ is a normal subgroup of G/N.

Let $\dfrac{G/N}{K/N}$ be the quotient group and let

$$\pi_2\colon G/N \to \frac{G/N}{K/N}$$

be the quotient map.
Let $\pi_1\colon G \to G/N$ be the quotient map so that

$$(\pi_1 \circ \pi_2)\colon G \xrightarrow{\pi_1} G/N \xrightarrow{\pi_2} \frac{G/N}{K/N}.$$

(a) Show that $\mathrm{im}(\pi_1 \circ \pi_2) = \dfrac{G/N}{K/N}$.
(b) Show that $\ker(\pi_1 \circ \pi_2) = K$.
(c) Using Theorem G.1.9(c), conclude that $G/K \simeq \dfrac{G/N}{K/N}$ as groups.

Exercise G.3.9. —
(a) Prove that if H and K are subgroups of a group G, then

$$HK = \{hk \mid h \in H, k \in K\}$$

is a subgroup of G if at least one of H and K is normal in G.
(b) Prove that if H and K are subgroups of a group G and K is normal in G then the subgroup

$$\langle H, K\rangle = HK.$$

Warning! Don't even think that $\mathrm{Card}(HK)$ has to be $\mathrm{Card}(H)\mathrm{Card}(K)$.
(c) Give an example of subgroups H and K of a group G such that K is normal and $\mathrm{Card}(HK) \neq \mathrm{Card}(H)\mathrm{Card}(K)$.

Exercise G.3.10. — Let G be a group. Let K be a normal subgroup of G and let H be a subgroup of G. Let

$$\begin{array}{rccc} \pi\colon & H & \to & G/K \\ & h & \mapsto & hK \end{array}$$

be the restriction to H of the quotient map $\pi\colon G \to G/K$.

(a) Show that $\ker \pi = H \cap K$.

(b) Show that $\mathrm{im}\pi = \dfrac{HK}{K} = \{hK \mid h \in H\}$.

(c) Using Theorem G.1.9, conclude that $\dfrac{H}{H \cap K} \simeq \dfrac{HK}{K}$.

Exercise G.3.11. — Let H_1 and H_2 be subgroups of groups G_1 and G_2, respectively.

(a) Show that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

(b) Let $\pi_1\colon G_1 \to G_1/H_1$ and $\pi_2\colon G_2 \to G_2/H_2$ be the quotient maps. Define a function

$$\begin{array}{rccc} (\pi_1 \times \pi_2)\colon & G_1 \times G_2 & \to & G_1/H_1 \times G_2/H_2 \\ & (g_1, g_2) & \mapsto & (g_1H_1, g_2H_2). \end{array}$$

Show that $\pi_1 \times \pi_2$ is a well defined surjective group homomorphism.

(c) Show that $\ker(\pi_1 \times \pi_2) = H_1 \times H_2$.

(d) Using Theorem G.1.9, conclude that

$$\frac{G_1 \times G_2}{H_1 \times H_2} \simeq \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$