G.4. Exercises: Group actions

Exercise G.4.1. — Let G be a group. G acts on itself by left multiplication.

$$\begin{array}{ccc} G\times G & \to & G \\ (g,h) & \mapsto & gh \end{array}$$

(a) Show that there is a single orbit, G.
(b) Show that if $h\in G$ the stabilizer of h is $\{1\}$.

Exercise G.4.2. — Prove the following theorem

(Cayley's Theorem) *Let G be a finite group and let $n=\mathrm{Card}(G)$. Then G is isomorphic to a subgroup of the symmetric group S_n.*

by completing the steps below:

(a) Show that

$$\begin{array}{cccc} p\colon & G & \to & \{\text{functions from } G \text{ to } G\} \\ & g & \longmapsto & \sigma_g \end{array} \quad \text{where} \quad \begin{array}{cccc} \sigma_g\colon & G & \to & G \\ & h & \mapsto & gh \end{array}$$

is a function.
(b) Show that $\mathrm{im}\, p=\{\text{bijective functions from } G \text{ to } G\}$, and identify $\{\text{bijective functions from } G \text{ to } G\}$ with S_n.
(c) Show that $p\colon G\to S_n$ as given by (a) and (b) is a group homomorphism.
(d) Show that $p\colon G\to S_n$ is injective and conclude that p is an isomorphism.

Exercise G.4.3. — Let G be a finite group and let $n=\mathrm{Card}(G)$. Let S_n denote the symmetric group on n. For each $g\in G$ define a map

$$\begin{array}{cccc} m_g\colon & G & \to & G \\ & h & \mapsto & gh. \end{array}$$

(a) Show that we can view $m_g\in S_n$ as a permutation of the elements of G.
(b) Show that

$$\text{if } g_1,g_2\in G \quad \text{then} \quad m_{g_1}\circ m_{g_2}=m_{g_1g_2},$$

since $m_{g_1}\big(m_{g_2}(h)\big)=m_{g_1}(g_2h)=g_1g_2h=m_{g_1g_2}(h)$.
(c) Show that if $1\in G$ denotes the identity in G then m_1 is the identity map on G.
(d) Show that if $g\in G$ then $m_{g^{-1}}$ is the inverse of the map m_g.
(e) Show that if $g,h\in G$ and $m_g=m_h$ then $g=m_g(1)=m_h(1)=h$.

Define a map

$$\begin{array}{cccc} \varphi\colon & G & \to & S_n \\ & g & \mapsto & m_g. \end{array}$$

(f) Show that, by (b) above, φ is a homomorphism.
(g) Show that, by (e), φ is injective.
(h) Using Theorem 1.1.15 c), conclude that

$$G\simeq \mathrm{im}\varphi\subseteq S_n.$$

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Exercise G.4.4. — Let H be a subgroup of a group G. The group H acts on G by right multiplication.

$$\begin{array}{ccc} H \times G & \to & G \\ (h,g) & \mapsto & gh^{-1}. \end{array}$$

(a) Show that if $g \in G$, the orbit of g under this action is the coset gH. Thus, the orbits are the cosets G/H.
(b) Show that the stabilizer of an element $g \in G$ is the group $\{1\}$.
(c) Using Proposition 1.2.4,REFERENCE give another proof of Proposition 1.1.3.REFERENCE
(d) Using Corollary 1.2.7,REFERENCE show that

$$\mathrm{Card}(H) = \mathrm{Card}(gH)\mathrm{Card}(\{1\}) = \mathrm{Card}(gH),$$

and give another proof of Proposition 1.1.4.REFERENCE

Exercise G.4.5. — Let H be a subgroup of a group G. The group G acts on G/H by left multiplication.

$$\begin{array}{ccc} G \times G/H & \to & G/H \\ (g,kH) & \mapsto & gkH \end{array}$$

(a) Show that there is one orbit under this action, G/H.
(b) Show that the stabilizer of the identity coset H is H and the stabilizer of a coset kH for $k \in G$ is the group kHk^{-1}.
(c) Use Corollary 1.2.7 REFERENCE to show that

$$\mathrm{Card}(G) = \mathrm{Card}(G/H)\mathrm{Card}(H),$$

and thus give another proof of Corollary 1.1.5.REFERENCE

Exercise G.4.6. — Let $p \in \mathbb{Z}_{>0}$ be a prime. A **p-group** is a group of cardinality p^a with $a \in \mathbb{Z}_{>0}$. Let G be a p-group.

(a) Show that G contains an element of order p by showing that if $x \in G$ and x has order p^b then $g = p^{b-1}$ has order p.
(b) Use the class equation (Proposition G.2.10),

$$\mathrm{Card}(G) = \mathrm{Card}(Z(G)) + \sum_{\mathrm{Card}(\mathcal{C}_{g_i})>1} \mathrm{Card}(\mathcal{C}_{g_i}),$$

to show that if $\mathrm{Card}(G) \neq 1$ then $Z(G) \neq \{1\}$.
(c) Show that there exists a chain of normal subgroups of G Use part (a) to show that G/N_i contains an element of order p, Then show that G/N_i contains a normal subgroup $\mathcal{H}$ of order p and use the correspondence ??? to conclude that G contains a normal subgroup N_{i+1} such that $N_{i+1}/N_i = H$. Use the fact that $Z(G) \neq \{1\}$ to start the induction. Conclude that if $\mathrm{Card}(G) = p^a$ then there exists a chain of normal subgroups of G,

$$\{1\} \subseteq N_1 \subseteq \cdots N_{a-1} \subseteq G \qquad \text{such that} \quad \mathrm{Card}(N_i) = p^i.$$

Exercise G.4.7. — Let G be a group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime.

(a) Show that if $\mathrm{Card}(G) = p$ then $G \cong \mathbb{Z}/p\mathbb{Z}$.
(b) Show that if $\mathrm{Card}(G) = p^2$ then $G \cong \mathbb{Z}/p^2\mathbb{Z}$ or $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Exercise G.4.8. — **First Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A **p-Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that G has a p-Sylow subgroup by completing the following steps.

(a) Let $\Lambda^{p^a}(G)$ be the set of subsets of G of cardinality p^a. Show that if $j \in \{1, \ldots, p^a\}$ and p^i divides $p^a b - j$ then p^i divides $p^a - j$. Conclude that
$$\mathrm{Card}(\Lambda^{p^a}(G)) = \binom{p^a b}{p^a} \quad \text{is not divisible by } p.$$

(b) Consider the action of G on $\Lambda^{p^a}(G)$ by left multiplication and use Proposition **??**,
$$\mathrm{Card}(\Lambda^{p^a}(G)) = \sum_{\text{distinct orbits}} \mathrm{Card}(GS),$$
to conclude that there exists $S \in \Lambda^{p_a}(G)$ such that the cardinality of the orbit of S is not divisible by p.

(c) Let $P = \mathrm{Stab}_G(S)$ and show that $\mathrm{Card}(P) = p^a$.

Exercise G.4.9. — **Second Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A **p-Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that all p-Sylow subgroups of G are conjugate by completing the following steps.

(a) Let P and H be p-Sylow subgroups of G. Let H act on G/P by left multiplication. Use
$$\mathrm{Card}(G/P) = \sum_{\text{distinct orbits}} \mathrm{Card}(HgP),$$
to show that there is an orbit HgP with $\mathrm{Card}(HgP) = 1$.

(b) Show that $H \subseteq gPg^{-1}$ and conclude that $H = gPg^{-1}$.

Exercise G.4.10. — **Third Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A **p-Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that the number of p-Sylow subgroups of G is 1 mod p by completing the following steps.

(a) Let P be a p-Sylow subgroup of G. Let P act on the set $\mathcal{S}$ of p-Sylow subgroups of G by conjugation. Show that if $P*Q$ is an orbit under this action then $\mathrm{Card}(P*Q) = 1$ or p divides $\mathrm{Card}(P*Q)$.

(b) Assume $\mathrm{Card}(P*Q) = 1$ and let $N(Q)$ be the normalizer of Q. Show that both P and Q are both p-Sylow subgroups of $N(Q)$.

(c) Assume $\mathrm{Card}(P*Q) = 1$. Use the second Sylow theorem and part (b) to show that $P = Q$.

(d) Use part (a) and (c) and
$$\mathrm{Card}(\mathcal{S}) = \sum_{\text{distinct orbits}} \mathrm{Card}(P*Q)$$
to conclude that $\mathrm{Card}(\mathcal{S}) = 1 \bmod p$.

Exercise G.4.11. — **Fourth Sylow theorem.** Let G be a finite group. Let $p \in \mathbb{Z}_{\geqslant 0}$ be a prime. Write $\mathrm{Card}(G) = p^a b$ where b is not divisible by p. A **p-Sylow subgroup** of G is a subgroup of G of cardinality p^a. Show that the number of p-Sylow subgroups divides $\mathrm{Card}(G)$ by completing the following steps.

(a) Let G act on the set $\mathcal{P}$ of p-Sylow subgroups of G by conjugation. Use the second Sylow theorem to conclude that there is only one orbit under this action.
(b) Conclude, from (a), that the number of p-Sylow subgroups divides $\mathrm{Card}(G)$.