Lecture 7 Calculus 2

An even function is a function $f: \mathbb{R} \to \mathbb{R}$ such that if $x \in \mathbb{R}$ then $f(x) = f(-x)$

An odd function is a function $f: \mathbb{R} \to \mathbb{R}$ such that if $x \in \mathbb{R}$ then $f(x) = -f(x)$

The hyperbolic sine and hyperbolic cosine are

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

Define

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Example 2.0 If $x = \cosh t$ and $y = \sinh t$ then

$$x^2 - y^2 = 1$$

Example 2.1 Assume $t \in \mathbb{R}_{<0}$ and $\cosh t = \frac{13}{12}$.

Then

$$\sinh^2 t = \cosh^2 t - 1 = \left(\frac{13}{12}\right)^2 - 1 = \frac{13^2 - 12^2}{12^2} = \frac{5^2}{12^2}$$

So $\sinh t = \pm \frac{5}{12}$

Since $t \in \mathbb{R}_{<0}$ and $\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$

then $\sinh t \in \mathbb{R}_{<0}$ and

$$\sinh t = -\frac{5}{12}$$

Example 2.2 Show that

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

Solution:

$$\sinh x \cosh y + \cosh x \sinh y$$

$$= \frac{1}{2}(e^{x} - e^{-x})\frac{1}{2}(e^{y} + e^{-y}) + \frac{1}{2}(e^{x} + e^{-x})\frac{1}{2}(e^{y} - e^{-y})$$

$$= \frac{1}{4}(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y})$$

$$+ \frac{1}{4}(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})$$

$$= \frac{1}{4}(e^{x+y} + e^{x+y} - e^{-(x+y)} - e^{-(x+y)})$$

$$= \frac{1}{4}(2e^{x+y} - 2e^{-(x+y)}) = \frac{1}{2}(e^{x+y} - e^{-(x+y)}).$$

$$= \sinh(x+y) \;//.$$

Example 2.3 Show that $\frac{d}{dx}(\cosh x) = \sinh x$

Solution: $$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{1}{2}(e^{x} + e^{-x})\right)$$

$$= \frac{1}{2}(e^{x} + e^{-x}(-1)) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh x \;//.$$

Example 2.4 Let $y = \sqrt{\sinh(6x)}$. Find $\frac{dy}{dx}$.

Solution: $$\frac{dy}{dx} = \frac{d\sqrt{\sinh(6x)}}{dx} = \frac{d}{dx}\left((\sinh(6x))^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}(\sinh(6x))^{-\frac{1}{2}}\cosh(6x)\cdot 6$$

$$= \frac{3\cosh(6x)}{\sqrt{\sinh(6x)}} = \frac{1}{y}\,3\cosh(6x). \,//$$

Example 2.5 Show that

$$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1})$$

Solution To show: $\operatorname{arcsinh} x = \log(x+\sqrt{x^2+1})$.

To show: $x = \sinh(\log(x+\sqrt{x^2+1}))$

$$\sinh(\log(x+\sqrt{x^2+1})) = \frac{1}{2}\left(e^{\log(x+\sqrt{x^2+1})} - e^{-\log(x+\sqrt{x^2+1})}\right)$$

$$= \frac{1}{2}\left(x+\sqrt{x^2+1} - \frac{1}{e^{\log(x+\sqrt{x^2+1})}}\right)$$

$$= \frac{1}{2}\left(x+\sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+1}}\right) = \frac{1}{2}\left(\frac{(x+\sqrt{x^2+1})^2 - 1}{x+\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+2x\sqrt{x^2+1}+x^2+1-1}{x+\sqrt{x^2+1}}\right) = \frac{1}{2}\left(\frac{2x(x+\sqrt{x^2+1})}{x+\sqrt{x^2+1}}\right) = x. \,//$$

Example 2.6 Simplify $\cosh(\operatorname{arctanh} x)$.

Solution: Let $y = \cosh(\operatorname{arctanh} x)$.

Then $\operatorname{arccosh} y = \operatorname{arctanh} x$.

So $\tanh(\operatorname{arccosh} y) = x$

So $\dfrac{\sinh(\operatorname{arccosh} y)}{\cosh(\operatorname{arccosh} y)} = x$.

So $\dfrac{\sinh(\operatorname{arccosh} y)}{y} = x$. Let $z = \operatorname{arccosh} y$.

So $\sinh z = x \cosh z$

So $\frac{1}{2}(e^{z} - e^{-z}) = x\,\frac{1}{2}(e^{z} + e^{-z})$

So $(1-x)e^{z} = (1+x)e^{-z}$

So $\dfrac{(1+x)}{(1-x)} = e^{2z}$

So $z = \frac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$

So $\operatorname{arcosh} y = \frac{1}{2}\log\left(\dfrac{1+x}{1-x}\right) = \log\left(\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}\right)$

So
$$y = \frac{1}{2}\left(\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} + \left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} + \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(\frac{1+x+1-x}{\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}}.$$

Example 2.7 Prove that $\frac{d}{dx}(\text{arcsinh}\, x) = \frac{1}{\sqrt{x^2+1}}$.

Solution: Let $y = \text{arcsinh}\, x$.

Then $\sinh y = x$.

So $\cosh y \frac{dy}{dx} = \frac{dx}{dx} = 1$. So $\frac{dy}{dx} = \frac{1}{\cosh y}$.

Left hand side: $\frac{d(\text{arcsinh}\, x)}{dx} = \frac{dy}{dx} = \frac{1}{\cosh y}$.

Righthand side:

$$\frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{(\sinh y)^2+1}} = \frac{1}{\sqrt{(\cosh y)^2}} = \frac{1}{\cosh y}.$$

//

Example 2.8 Find $\frac{d}{dx}(\text{arctanh}(2x)\cosh(3x))$

Solution

$$\frac{d}{dx}(\text{arctanh}(2x)\cosh(3x))$$

$$= \text{arctanh}(2x) \cdot \sinh(3x)3 + \left(\frac{d}{dx}\text{arctanh}(2x)\right)\cosh(3x)$$

$$= 3\,\text{arctanh}(2x)\sinh(3x) + \left(\frac{1}{1-(2x)^2}\right) \cdot 2 \cdot \cosh(3x)$$

$$= 3\,\text{arctanh}(2x)\sinh(3x) + \frac{2\cos(3x)}{1-4x^2}$$