Example 1.17 Evaluate

$$\lim_{x\to 0} \log(7+3x-x^2).$$

Solution: $\lim_{x\to 0} \log(7+3x-x^2) = \log(7+3\cdot 0-0^2) = \log 7$

because $\lim_{x\to 0} 7+3x-x^2 = 7$ and

$\log y$ is continuous at $y=7$.

Example 1.17 Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Solution: $\displaystyle \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0} \frac{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots}{x}$

$\displaystyle = \lim_{x\to 0} 1-\frac{1}{3!}x^2+\frac{1}{5!}x^5+\cdots = 1+0+0+\cdots = 1.$

Alternatively, use L'Hopital's rule:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\sin x}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1.$$

Example 1.18 Evaluate $\displaystyle \lim_{x\to\infty}\frac{3x^2-2x+3}{x^2+4x+4}$

Solution: $\displaystyle \lim_{x\to\infty}\frac{3x^2-2x+3}{x^2+4x+4} = \lim_{x\to\infty}\frac{(3x^2-2x+3)\frac{1}{x^2}}{(x^2+4x+4)\frac{1}{x^2}}$

$\displaystyle = \lim_{x\to\infty}\frac{3-\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}} = \cancel{\lim} \frac{3-0+0}{1+0+0} = \frac{3}{1} = 3.$

Example 1.19 Evaluate $\lim_{x\to\infty} x^{-1/3}\log x$.

Solution:

$$\lim_{x\to\infty} x^{-1/3}\log x = \lim_{y\to\infty} (e^y)^{-\frac{1}{3}}\log e^y$$

$$= \lim_{y\to\infty} e^{-\frac{1}{3}y}\, y = \lim_{y\to\infty} \frac{y}{e^{1/3\, y}} = \lim_{y\to\infty} \frac{y\cdot\frac{1}{y}}{e^{\frac{1}{3}y}\cdot\frac{1}{y}}$$

$$= \lim_{y\to\infty} \frac{1}{\left(1+\frac{1}{3}y+\frac{1}{2!}\left(\frac{1}{3}y\right)^2+\cdots\right)\frac{1}{y}}$$

$$= \lim_{y\to\infty} \frac{1}{\frac{1}{y}+\frac{1}{3}+\frac{1}{2}\left(\frac{1}{3}\right)^2 y+\frac{1}{3!}\left(\frac{1}{3}\right)^3 y^2+\cdots}$$

$$\le \lim_{y\to\infty} \frac{1}{\frac{1}{2}\left(\frac{1}{3}\right)^2 y} = \lim_{y\to\infty} 18\,\frac{1}{y} = 18\cdot 0 = 0.$$

Example 1.22 Determine whether the following sequences converge or diverge:

(a) $a_n = \frac{1}{n}$, (b) $a_n = (-1)^{n-1}$, (c) $a_n = n$.

Solution:

(a) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ converges to 0 in $\mathbb{R}$

(b) $(1, -1, 1, -1, \ldots)$ does not converge in $\mathbb{R}$

(c) $(1, 2, 3, \ldots)$ does not converge in $\mathbb{R}$.

Example 1.21 Let $a \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions. Assume

$\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists.

Show that

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

Proof Let $L = \lim_{x \to a} f(x)$ and $M = \lim_{x \to a} g(x)$.

To show: $\lim_{x \to a} (f(x) + g(x)) = L + M$.

To show: If $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that

if $d(x, a) < \delta$ then $d(f(x) + g(x), L + M) < \varepsilon$.

Assume $\varepsilon = 10^{-k}$ (you want k decimal places accuracy)

Let l_1 be such that

if $d(x, a) < 10^{-l_1}$ then $d(f(x), L) < 10^{-(k+1)}$

Let l_2 be such that

if $d(x, a) < 10^{-l_2}$ then $d(f(x), M) < 10^{-(k+1)}$

Let $\delta = 10^{-(l_1 + l_2 + 1)}$ ($l_1 + l_2 + 1$ digits accuracy on the dials).

To show: If $d(x,a)<\delta$ then

$$d(f(x)+g(x), L+M)<\varepsilon.$$

Assume $d(x,a)<\delta$. So $|x-a|<10^{-(k_1+k_2+1)}$

To show: $d(f(x)+g(x), L+M)<\varepsilon$.

$$\begin{aligned} d(f(x)+g(x), L+M) &= |f(x)+g(x)-(L+M)| \\ &= |f(x)-L+g(x)-M| \\ &= |f(x)-L|+|g(x)-M| \\ &< 10^{-(k+1)}+10^{-(k+1)} = \frac{2}{10}10^{-k} = \frac{1}{5}10^{-k} < 10^{-k} = \varepsilon. \end{aligned}$$