# Calculus 2 Last Lecture

25.10.2019 ①
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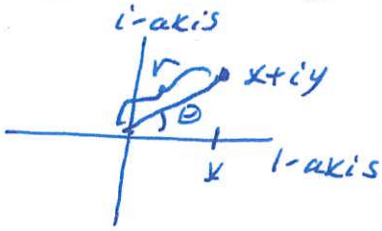
## Complex numbers:

$$i^2=-1, \quad \overline{x+iy} = x-iy, \quad |x+iy| = \sqrt{x^2+y^2}$$

$$x+iy = re^{i\theta}$$




with $r = \sqrt{x^2+y^2}$ $\qquad x = r\cos\theta$

$\tan\theta = \frac{y}{x}$ $\qquad y = r\sin\theta$

## Trig and hyperbolic functions

$$\sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right) \qquad \cos x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

$$\sinh x = \frac{1}{2}\left(e^{x}-e^{-x}\right) \qquad \cosh x = \frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$e^{ix} = \cos x + i\sin x \qquad e^{x} = \sinh x + \cosh x$$

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1.$$

## Limits, sequences, series

$L = \lim_{x\to a} f(x)$ means $f(x)$ gets closer and closer to $L$ as $x$ gets closer and closer to $a$

$$\frac{1}{1-s} = 1+s+s^2+s^3+\cdots \quad \text{(geometric series)}$$

$$e^s = 1+s+\frac{1}{2}s^2+\frac{1}{3!}s^3+\frac{1}{4!}s^4+\cdots \quad \text{(exponential series)}$$

$$\zeta(s) = 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

(harmonic series $\zeta(1)$
p-series $\zeta(p)$ for $p \in \mathbb{R}_{>0}$
Riemann zeta function).

Continuity The function $f$ is continuous if

$$\lim_{x\to a} f(g) = f(\lim_{x\to a} g)$$

and $f(x)=x^n$ and $f(x)=e^x$ are continuous.

Differentiation and Integration $\int \frac{df}{dx}\,dx = f$

$$\frac{d(cf)}{dx} = c\frac{df}{dx} \qquad \int (cg)\,dx = c\int g\,dx$$

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} \qquad \int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v \qquad \int u\,dv = uv - \int v\,du$$

$$\frac{d(f\circ u)}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \int g\frac{du}{dx}\,dx = \int g\,du$$

Fundamental theorems

$$Df(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad \text{(first principles)}$$

If $\frac{df}{dx} = g$ then $\left.\frac{df}{dx}\right]_{x=a} = g(a)$

Fundamental theorem of change:

$$Df(a) = \left.\frac{df}{dx}\right]_{x=a}.$$

$$\int_{x=a}^{x=b} g\,dx = \lim_{N\to\infty}\left(g(a)\tfrac{1}{N} + g(a+\tfrac{1}{N})\tfrac{1}{N} + \dots + g(b-\tfrac{1}{N})\cdot\tfrac{1}{N}\right)$$

If $\frac{df}{dx} = g$ then let $\int g\,dx\Big]_{x=a}^{x=b} = f(b)-f(a)$.

Fundamental theorem of calculus:

$$\int_{x=a}^{x=b} g\,dx = \int g\,dx\Big]_{x=a}^{x=b}.$$

<u>Differential Equations</u>

Separable: If $\frac{dy}{dx} = PQ$ then $\frac{1}{P}\frac{dy}{dx} = Q$

and $\int \frac{1}{P}\,dy = \int Q\,dx$

Integrating factors: If $Q = \frac{dy}{dx} + Py = \frac{dy}{dx} + \frac{1}{I}\frac{dI}{dx}y$

then $QI = I\frac{dy}{dx} + \frac{dI}{dx}y = \frac{d(Iy)}{dx}$

so that $\int QI\,dx = Iy$ and $y = \frac{1}{I}\int QI\,dx$

with $\log I = \int P\,dx$.

English to math translation:

Population models, mixing problems

springs, circuits

## Derivative matrix and chain rule

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \longmapsto \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$D_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Chain rule: $D_{f\circ u} = D_f D_u$.

Functions $f : \mathbb{R}^2 \to \mathbb{R}$. Let $(x_0, y_0) \in \mathbb{R}^2$.

Gradient at $(x_0, y_0)$: $\vec{\nabla} f \big|_{(x,y)=(x_0,y_0)} = \left( \frac{\partial f}{\partial x}\Big|_{(x,y)=(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x,y)=(x_0,y_0)} \right)$

Hessian at $(x_0, y_0)$

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\Bigg|_{(x,y)=(x_0,y_0)}$$

Directional derivative in direction $\hat{u}$ at $(x_0, y_0)$

$$\vec{\nabla} f \big|_{(x,y)=(x_0,y_0)} \cdot \hat{u}.$$

Linear approximation to $f$ at $(x_0, y_0)$

$$f(x,y) \cong f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x,y)=(x_0,y_0)} (x-x_0) + \frac{\partial f}{\partial y}\Big|_{(x,y)=(x_0,y_0)} (y-y_0)$$

Tangent plane to $f$ at $(x_0, y_0)$

$$z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x,y)=(x_0,y_0)} (x-x_0) + \frac{\partial f}{\partial y}\Big|_{(x,y)=(x_0,y_0)} (y-y_0).$$

## Stationary points, local minima, local maxima, and Saddle points

A <u>stationary point</u> is $(x_0, y_0)$ with

$$\vec{\nabla} f \Big|_{(x,y)=(x_0,y_0)} = (0,0)$$

local minimum

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(x_0,y_0)} > 0$$

$$\det(H_f) > 0$$

local maximum

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(x_0,y_0)} < 0$$

$$\det(H_f) > 0$$

saddle point

$$\det(H_f) < 0$$

Calc 2 Last Lecture 25.10.2019 1
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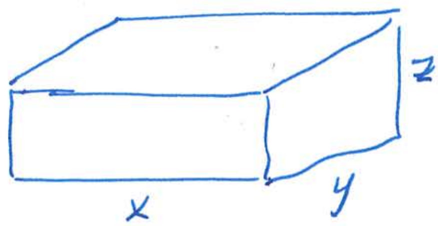
Consider an open top box of
length $x$, width $y$ and height $z$.
with volume 4. Find the minimum amount
of metal to make the box.

Solution




Volume $= xyz = 4$.

Minimize metal, $M = 2xz + 2yz + xy$

Since $4 = xyz$ then $z = \frac{4}{xy}$.

So $M = 2x\frac{4}{xy} + 2y\frac{4}{xy} + xy = \frac{8}{y} + \frac{8}{x} + xy$.

Then
$$\vec{\nabla} M = \left(\frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}\right) = \left(0 - \frac{8}{x^2} + y, \frac{-8}{y^2} + 0 + x\right)$$
$$= \left(y - \frac{8}{x^2}, x - \frac{8}{y^2}\right).$$

The stationary point is when
$(0,0) = \vec{\nabla} M = \left(y - \frac{8}{x^2}, x - \frac{8}{y^2}\right)$ so that
$y = \frac{8}{x^2}$ and $x = \frac{8}{y^2}$ giving $y = \frac{8}{\left(\frac{8}{y^2}\right)^2} = \frac{8y^4}{8^2}$

So $\frac{1}{8}y^4 - y = 0$. So $y^4 - 8y = 0$. So $y(y^3 - 8) = 0$.
So $y = 0$ or $y^3 = 8$. So $y = 0$ or $y = 2$.

If $y=2$ Calc 2 Last Lecture 25.10.2019 A. Ram ②

~~then~~ $x=\frac{8}{2^2}=\frac{8}{4}=2$ and $z=\frac{4}{xy}=\frac{4}{2\cdot 2}=1$

So $M=2\cdot2\cdot1+2\cdot2\cdot1+2\cdot2=\cancel{8+}\ 4+4+4=12$

~~If $y=0$ then~~ We want $y>0$.