Example 7.18 Let $f(x,y) = 4x^2+y^2$.

(a) Find ∇f at $(1,0)$ and $(0,2)$

(b) Show that ∇f is perpendicular to the level curves, by sketching ∇f at these points and the level curves of f.

Solution: If $(x,y)=(1,0)$ then $f(x,y)=4\cdot 1^2+0^2=4$

If $(x,y)=(0,2)$ then $f(x,y) = 4\cdot 0^2+2^2=4$.

So these are on the level curve

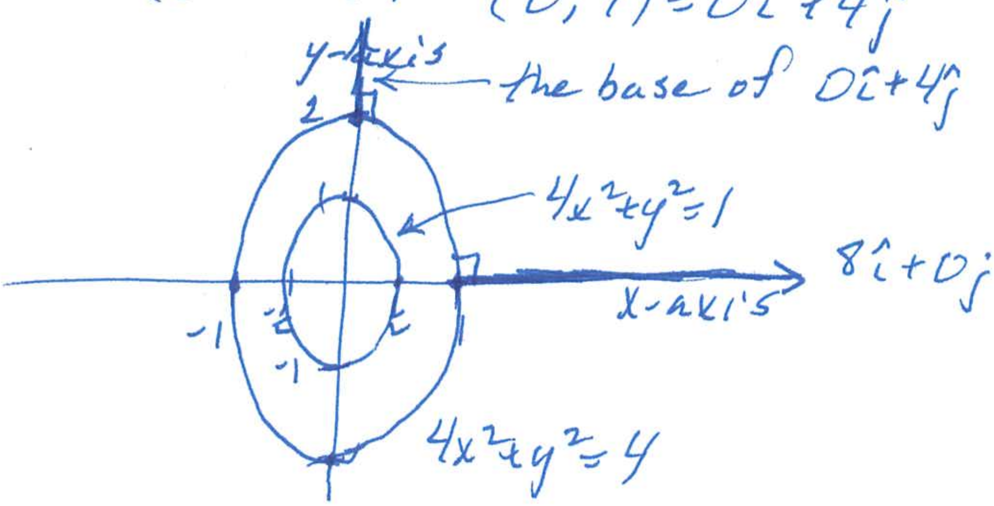
$$4x^2+y^2=4.$$

$$\nabla f\Big|_{(x,y)=(1,0)} = \left(\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y}\right)\Big|_{(x,y)=(1,0)} = (8x \ \ 2y)\Big|_{(x,y)=(1,0)}$$

$$= (8\cdot 1 \ \ 2\cdot 0) = (8 \ \ 0) = 8\hat{i}+0\hat{j}$$

$$\nabla f\Big|_{(x,y)=(0,2)} = \left(\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y}\right)\Big|_{(x,y)=(0,2)} = (8x \ \ 2y)\Big|_{(x,y)=(0,2)}$$

$$= (8\cdot 0 \ \ 2\cdot 2) = (0,4) = 0\hat{i}+4\hat{j}$$



Example 7.19 In what direction does $f(x,y)=xe^y$ increase/decrease most rapidly at $(2,0)$. Express direction as a unit vector.

Solution: $\nabla f = \left(\frac{\partial f}{\partial x} \ \ \frac{\partial f}{\partial y}\right) = \left(e^y \ \ xe^y\right)$

so

$$\nabla f\Big|_{(x,y)=(2,0)} = \left(e^0 \ \ 2\cdot e^0\right) = (1 \ \ 2) = \hat{\imath} + 2\hat{\jmath}.$$

The unit vector in the direction $\hat{\imath}+2\hat{\jmath}$ is

$$\frac{1}{\sqrt{1^2+2^2}}\cdot(\hat{\imath}+2\hat{\jmath}) = \frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{\jmath}$$

The direction of most rapid increase is

$$\frac{1}{\sqrt{5}}(1,2)$$

The direction of most rapid decrease is

$$\frac{-1}{\sqrt{5}}(1,2).$$

A stationary point is (x_0,y_0) such that

$$\nabla f\Big|_{(x,y)=(x_0,y_0)} = 0.$$

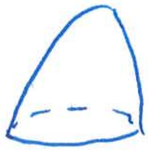
A critical point is (x_0,y_0) such that (x_0,y_0) is a stationary point

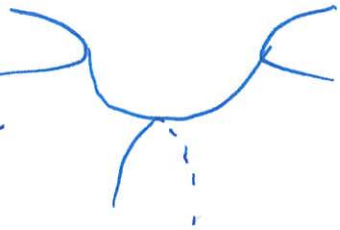
OR $\frac{\partial f}{\partial x}\Big|_{(x,y)=(x_0,y_0)}$ does not exist OR $\frac{\partial f}{\partial y}\Big|_{(x,y)=(x_0,y_0)}$ does not exist

If $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0;y_0)} > 0$ and $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\Bigg|_{(x,y)=(x_0,y_0)} > 0$

then  local minimum.

If $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(x_0,y_0)} < 0$ and $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\Bigg|_{(x,y)=(x_0,y_0)} > 0$

then  local maximum.

If $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} < 0$ then 

saddle point.

<u>Example 7.10</u> Find and classify the stationary points of $f(x,y) = x^3+y^3+3x^2-3y^2-8$.

<u>Solution:</u> $\nabla f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right) = (3x^2+6x \quad 3y^2-6y)$

If $0 = \nabla f\Big|_{(x,y)=(x_0,y_0)} = (3x_0^2+6x_0 \quad 3y_0^2-6y_0)$

$= (3x_0(x_0+2) \quad 3y_0(y_0-2))$ then

$(x_0,y_0) = (0,0)$ <u>OR</u> $(x_0,y_0) = (-2,0)$

<u>OR</u> $(x_0,y_0) = (0,2)$ <u>OR</u> $(x_0,y_0) = (-2,2)$

These are the stationary points.

<u>The point $(x_0,y_0) = (0,0)$</u>

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(0,0)} = 6x+6\Big|_{(x,y)=(0,0)} = 6\cdot 0+6 = 6 > 0$$

and

$$\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \det\begin{pmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{pmatrix}\Bigg|_{(x,y)=(0,0)}$$

$$= (6x+6)(6y-6)\Big|_{(x,y)=(0,0)} = 6(-6) = -36 < 0.$$

So $(x_0,y_0) = (0,0)$ is a saddle point.

<u>The point $(x_0, y_0) = (-2, 0)$</u>

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(-2,0)} = \left. 6x+6 \right|_{(x,y)=(-2,0)} = 6(-2)+6 = -6 < 0$$

and

$$\det\left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_{(x,y)=(-2,0)} = \det\left.\begin{pmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{pmatrix}\right|_{(x,y)=(-2,0)}$$

$$= \left.(6x+6)(6y-6) - 0\cdot 0\right|_{(x,y)=(-2,0)} = (6(-2)+6)(6\cdot 0 - 6)$$

$$= (-6)(-6) = 36 > 0$$

so $(x_0, y_0) = (-2, 0)$ is a local maximum.

<u>The point $(x_0, y_0) = (0, 2)$</u>

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(0,2)} = \left. 6x+6 \right|_{(x,y)=(0,2)} = 6\cdot 2+6 = 18 > 0$$

and

$$\det\left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_{(x,y)=(0,2)} = \det\left.\begin{pmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{pmatrix}\right|_{(x,y)=(0,2)}$$

$$= \left.(6x+6)(6y-6)\right|_{(x,y)=(0,2)} = 6(6(+2)-6) = 6(\overset{6}{\cancel{-18}}) > 0.$$

So $(x_0, y_0) = (0, 2)$ is a local minimum.

The point $(x_0, y_0) = (-2, 2)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(-2,2)} = \left.6x+6\right|_{(x,y)=(-2,2)} = 6(-2)+6 = -6 < 0$$

and

$$\det\left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_{(x,y)=(-2,2)} = \det\left.\begin{pmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{pmatrix}\right|_{(x,y)=(-2,2)}$$

$$= \left.(6x+6)(6y-6)\right|_{(x,y)=(-2,2)} = (-6)\cdot 6 = -36 < 0.$$

So $(x_0, y_0) = (-2, 2)$ is a saddle point.

Example 7.21 Find and classify the stationary points of $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = y\sin x$

Solution $\nabla f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right) = (y\cos x \quad \sin x)$

If $0 = \nabla f\Big|_{(x,y)=(x_0,y_0)} = (y_0\cos x_0 \quad \sin x_0)$

then $x_0 \in \{\ldots -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$ and $y_0 = 0$.

The point $(x_0, y_0) = (k\pi, 0)$ where $k \in \mathbb{Z}$.

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(k\pi,0)} = -y\sin x\Big|_{(x,y)=(k\pi,0)} = -0\sin k\pi = 0$$

and

$$\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\Bigg|_{(x,y)=(k\pi,0)} = \det\begin{pmatrix} -y\sin x & \cos x \\ \cos x & 0 \end{pmatrix}\Bigg|_{(x,y)=(k\pi,0)}$$

$$= (-y\sin x \cdot 0 - \cos x \cdot \cos x)\Big|_{(x,y)=(k\pi,0)}$$

$$= -\cos^2 x\Big|_{(x,y)=(k\pi,0)} = -\cos^2(k\pi) = -1. < 0.$$

So $(x_0, y_0) = (k\pi, 0)$ is a saddlepoint.