Calculus 2 Lecture 29

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Let $f:\mathbb{R}^2 \to \mathbb{R}$. Let $(x_0, y_0) \in \mathbb{R}$

(Let $d(p,q)$ = distance from p to q.)

The limit of f as (x,y) approaches (x_0,y_0) is $L \in \mathbb{R}$ such that

if $k \in \mathbb{Z}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $(a,b) \in \mathbb{R}^2$ and $d((a,b),(x_0,y_0)) < 10^{-\ell}$

then $d(f(a,b), L) < 10^{-k}$.

Write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ if } L \text{ exists.}$$

The function f is continuous at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

The partial derivative of f with respect to x at (x_0,y_0) from first principles is

$$f_x(x_0,y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

The partial derivative of f with respect to y at (x_0,y_0) from first principles is

$$f_y(x_0,y_0) = \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}.$$

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Interpret

$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)}$ as the slope of f at the point (x_0,y_0) in the direction of increasing x, and

$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)}$ as the slope of f at the point (x_0,y_0) in the direction of increasing y.

Then, near (x_0,y_0),

$$f(x,y) \approx f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)} (y-y_0)$$

is the <u>linear approximation to f at (x_0,y_0)</u>

and

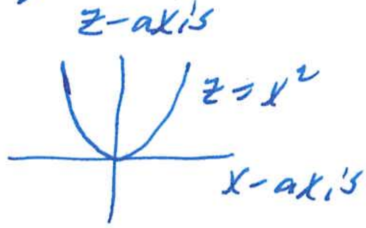
with $z_0 = f(x_0,y_0)$,

the equation of the <u>tangent plane to f at (x_0,y_0)</u> is

$$z - z_0 = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)} (y-y_0)$$

Example 7.5 For which values of x and y is $f(x,y) = x^2 + y^2$?

Solution Since x^2 is continuous for all (x,y) and y^2 is continuous for all (x,y) then x^2+y^2 (a polynomial in x and y) is continuous for all $(x,y) \in \mathbb{R}^2$. //



Example 7.6 Evaluate $\lim_{(x,y)\to(2,1)} \log(1+2x^2+3y^2)$

Solution

$$\lim_{(x,y)\to(2,1)} \log(1+2x^2+3y^2)$$

$$= \log\left(\lim_{(x,y)\to(2,1)} (1+2x^2+3y^2)\right)$$ since $f(z) = \log z$ is continuous for $z \in \mathbb{R}_{>0}$,

$$= \log(1+2\cdot 2^2+3\cdot 1^2),$$ since $1+2x^2+3y^2$, a polynomial, is continuous for $(x,y) \in \mathbb{R}^2$

$$= \log 12. \;//$$

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Example 7.7 Let $f(x,y) = xy^2$. Find $\frac{\partial f}{\partial y}$ from first principles.

Solution

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{x_0(y_0+h)^2 - x_0 y_0^2}{h} = \lim_{h\to 0} \frac{x_0 y_0^2 + 2x_0 y_0 h + x_0 h^2 - x_0 y_0^2}{h}$$

$$= \lim_{h\to 0} (2x_0 y_0 + x_0 h) = 2x_0 y_0 //.$$

Example 7.8 Let $f(x,y) = 3x^3y^2 + 3xy^4$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^3y^2 + 3xy^4) = \frac{\partial}{\partial x}(3x^3y^2) + \frac{\partial}{\partial x}(3xy^4)$$

$$= 9x^2y^2 + 3y^4$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^3y^2 + 3xy^4) = \frac{\partial}{\partial y}(3x^3y^2) + \frac{\partial}{\partial y}(3xy^4)$$

$$= 3x^3 \cdot 2y + 3x\,4y^3 = 6x^3y + 12xy^3. //$$

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<u>Example 7.9</u> Let $f(x,y) = y\log x + x\tanh(3y)$. Find f_x and f_y at $(1,0)$.

<u>Solution</u> $\frac{\partial f}{\partial x} = y\frac{1}{x} + 1\cdot\tanh(3y)$ and

$$\frac{\partial f}{\partial y} = 1\cdot\log x + x\,\text{sech}^2(3y)\cdot 3 = \log x + \frac{3x}{\cosh^2(3y)}.$$

So $f_x(1,0) = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,0)} = 0\cdot\frac{1}{1} + 1\cdot\tanh(3\cdot 0)$

$$= 0 + \frac{\sinh(0)}{\cosh(0)} = \frac{0}{1} = 0, \text{ and}$$

$$f_y(1,0) = \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,0)} = \log 1 + \frac{3\cdot 1}{\cosh^2(3\cdot 0)}$$

$$= 0 + \frac{3}{1^2} = 3.$$

<u>Example 7.10</u> Find the equation of the tangent plane to $z = f(x,y) = 2x^2 + y^2$ at $(1,1,3)$.

<u>Solution</u> The equation of the tangent plane is

$$z - 3 = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,1)}(x-1) + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,1)}(y-1)$$

$$= 4x\Big|_{(x,y)=(1,1)}(x-1) + 2y\Big|_{(x,y)=(1,1)}(y-1)$$

$= 4(x-1) + 2(y-1)$, so the equation is

$z = 4x - 4 + 2y - 2 + 3$ or $z = 4x + 2y - 3$