Example 6.6 Start with $y''+2y'-8y=0$. Let $D=\frac{d}{dx}$

Since $0=(D^2+2D-8)y=(D+4)(D-2)y$ then

$$y_H = c_1e^{-4x}+c_2e^{2x}, \text{ where } c_1 \text{ and } c_2 \text{ are constants}$$

are the solutions of the (homogeneous) equation

$y''+2y'-8y=0$.

(a) Solve $y''+2y'-8y=1-8x^2$

Put $y_p = ax^2+bx+c$ and solve for $a, b,$ and c.

$$y_p' = 2ax+b$$

$$y_p'' = 2a$$

If $1-8x^2 = y_p''+2y_p'-8y_p = 2a+2(2ax+b) - 8(ax^2+bx+c)$

$= -8ax^2+(4a-8b)x+2a+2b-8c$

then

$-8a=-8$

$4a-8b=0$ so that

$2a+2b-8c=1$

$a=1$

$b=\frac{4a}{8}=\frac{1}{2}$

$c=\frac{2a+2b-1}{8}=\frac{2+1-1}{8}=\frac{1}{4}$

So $y_p = x^2+\frac{1}{2}x+\frac{1}{4}$ and the general solution is

$$y = y_H+y_p = c_1e^{-4x}+c_2e^{2x}+x^2+\frac{1}{2}x+\frac{1}{4}$$

where c_1 and c_2 are constants.

(b) Solve $y'' + 2y' - 8y = e^{3x}$.

Put $y_p = ae^{3x}$ and solve for a.

$$y_p' = 3ae^{3x}$$
and
$$y_p'' = 9ae^{3x}$$

$$e^{3x} = y_p'' + 2y_p' - 8y_p = 9ae^{3x} + 2\cdot 3ae^{3x} - 8ae^{3x}$$
$$= (9a + 6a - 8a)e^{3x} = 7ae^{3x}, \text{ so } a = \frac{1}{7}.$$

and $y_p = \frac{1}{7}e^{3x}$. The general solution is

$$y = y_H + y_p = c_1 e^{-4x} + c_2 e^{2x} + \frac{1}{7}e^{3x}.$$

(c) Solve $y'' + 2y' - 8y = 85\cos x$.

Put $y_p = a\cos x + b\sin x$ and solve for a and b.

$$y_p' = -a\sin x + b\cos x$$
and
$$y_p'' = -a\cos x - b\sin x$$

$$85\cos x = y_p'' + 2y_p'' - 8y_p = -a\cos x - b\sin x$$
$$+ (-2a)\sin x + 2b\cos x$$
$$+ -8a\cos x - 8b\sin x$$
$$= (-a + 2b - 8a)\cos x + (-b - 2a - 8b)\sin x$$
$$= (-9a + 2b)\cos x + (-9b - 2a)\sin x.$$

So $-9a + 2b = 85$ and $-9b - 2a = 0$.

So $a = \frac{-9}{2}b$ and $-9\left(\frac{-9}{2}b\right) + 2b = 85$.

So $b = \dfrac{85 - \frac{81}{2}}{2} = \dfrac{\cancel{85}^{170} - 81}{4} = \dfrac{89}{4}$ and $a = \frac{-9}{2}\cdot\frac{89}{4}$

So $y_p = -\frac{9\cdot 89}{8}\cos x + \frac{89}{4}\sin x$ and the general solution is

$$y = y_H + y_p = c_1 e^{-4x} + c_2 e^{2x} + \frac{-9\cdot 89}{8}\cos x + \frac{89}{4}\sin x.$$

(d) Solve $y'' + 2y' - 8y = 3 - 24x^2 + 7e^{3x}$.

This is $y'' + 2y' - 8y = 3(1 - 8x^2) + 7e^{3x}$.

By (a)

$y_{p_1} = x^2 + \frac{1}{2}x + \frac{1}{4}$ is a solution of

$$y'' + 2y' - 8y = 1 - 8x^2, \text{ and}$$

$y_{p_2} = \frac{1}{7}e^{3x}$ is a solution of

$$y'' + 2y' - 8y = e^{3x}.$$

So $y'' + 2y' - 8y = 3(1 - 8x^2) + 7e^{3x}$ has general solution

$$\begin{aligned} y = y_H + 3y_{p_1} + 7y_{p_2} &= c_1 e^{-4x} + c_2 e^{2x} + 3\left(x^2 + \tfrac{1}{2}x + \tfrac{1}{4}\right) \\ &\quad + 7\left(\tfrac{1}{7}e^{3x}\right) \\ &= c_1 e^{-4x} + c_2 e^{2x} + e^{3x} + 3x^2 + \tfrac{3}{2}x + \tfrac{3}{4}, \end{aligned}$$

where c_1 and c_2 are constants.

Example 6.7 Solve $y''-y=e^x$.

Let $D=\frac{d}{dx}$ then $y''-y=0$ is

$0=(D^2-1)y=(D+1)(D-1)y$ which has solution

$y_H=c_1e^{-x}+c_2e^x$, where c_1 and c_2 are constants.

Let $y_p=axe^x$ and solve for a.

$$y_p'=a(xe^x+e^x)=axe^x+ae^x$$

$$y_p''=a(xe^x+e^x+e^x)=axe^x+2ae^x$$

and

$$e^x=y_p''-y_p=axe^x+2ae^x-(axe^x+ae^x)$$

$$=ae^x \text{ giving } a=1.$$

So $y_p=xe^x$ and the general solution is

$$y=y_H+y_p=c_1e^{-x}+c_2e^x+xe^x.$$

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A. Ram

Example 6.8 Solve $y''+2y'+y=e^{-x}$.

Let $D=\frac{d}{dx}$. Then $y''+2y'+y=0$ is

$0=(D^2+2D+1)y=(D+1)^2y$ which has

solution $y_H=c_1e^{-x}+c_2xe^{-x}$.

Let $y_p=ax^2e^{-x}$ and solve for a.

$$y_p'=a(x^2(-1)e^{-x}+2xe^{-x})=a(-x^2e^{-x}+2xe^{-x})$$

$$y_p''=a(-x^2(-e^{-x})-2xe^{-x}+2x(-e^{-x})+2e^{-x})$$

$$=a(x^2e^{-x}-4xe^{-x}+2e^{-x}).$$

Then
$$e^{-x}=y_p''+2y_p'+y_p=a(x^2e^{-x}-4xe^{-x}+2e^{-x})$$
$$+2a(-x^2e^{-x}+2xe^{-x})$$
$$+ax^2e^{-x}$$
$$=2ae^{-x} \text{ and } a=\tfrac{1}{2}.$$

So $y_p=\frac{1}{2}x^2e^{-x}$ and the general solution of $y''+2y'+y=e^{-x}$ is

$$y=y_H+y_p=c_1e^{-x}+c_2xe^{-x}+\tfrac{1}{2}x^2e^{-x}.$$