Limits

The accuracy set or tolerance set is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, 10^{-3}, \ldots\}$$

Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ (f is your machine that should produce the product ℓ)

Let $a \in X$ and $\ell \in \mathbb{R}$.

$\lim_{x \to a} f(x) = \ell$ means (f(x) gets closer and closer to ℓ as x gets closer and closer to a)

if $\varepsilon \in \mathbb{E}$ (if you give me a tolerance you want, i.e. a number of decimal places of accuracy you want)

then there exists (then my business, for a fee, tells you)

$\delta \in \mathbb{E}$ such that (the accuracy you need on the dials on your machine so that)

if $d(x,a) < \delta$ (if you get your dials within δ of a)

then $d(f(x), \ell) < \varepsilon$ (then the output of the machine will be within ε of exact)

Limits as $n \to \infty$

Let $X \subseteq \mathbb{R}$ and let $(a_1, a_2, a_3, \ldots)$ be a sequence in X.

Let $\ell \in \mathbb{R}$.

$\lim_{n\to\infty} a_n = \ell$ means (an gets closer and closer to ℓ as n gets larger and larger)

if $\varepsilon \in \mathbb{E}$ (if you give me a tolerance you want i.e. a number of decimal places of accuracy you want)

then there exists (then my business, for a fee, tells you)

$N \in \mathbb{Z}_{>0}$ such that (how many minutes you need to wait so that)

if $n \in \mathbb{Z}_{\geq N}$ (if you wait longer than N)

then $d(a_n, \ell) < \varepsilon$ (then the process has gotten within ε of exact)

Limit Theorems

(Addition theorem) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.
If $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists then $\lim_{x \to a} (f(x) + g(x))$ exists and

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

(Scalar multiplication theorem). Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. Let $c \in \mathbb{R}$.
If $\lim_{x \to a} f(x)$ exists then

$\lim_{x \to a} (cf(x))$ exists and $c \lim_{x \to a} f(x) = \lim_{x \to a} (cf(x))$

(Multiplication theorem) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.
If $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists then $\lim_{x \to a} (f(x) g(x))$ exists and

$$\lim_{x \to a} (f(x) g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right).$$

(Order theorem) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.

If f and g satisfy

if $x \in \mathbb{R}$ then $f(x) < g(x)$

and $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists

then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

(Composition of functions theorem)
Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$ and $\ell \in \mathbb{R}$.

If $\lim_{x \to a} g(x)$ exists and $\lim_{x \to a} f(g(x))$ exists and $\lim_{x \to a} g(x) = \ell$

then

$$\lim_{y \to \ell} f(y) = \lim_{x \to a} f(g(x)).$$

(Division theorem) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.

If $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists

and $\lim_{x \to a} g(x) \neq 0$

then $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right)$ exists and

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

<u>Power theorem</u> Let $n \in \mathbb{Z}_{>0}$. Let $a \in \mathbb{R}$

(a) $\lim_{x \to a} x^n = a^n$

(b) $\lim_{x \to a} x^{-n} = \begin{cases} a^{-n}, & \text{if } a \neq 0 \\ \text{does not exist}, & \text{if } a = 0 \end{cases}$

(c) $\lim_{x \to a} x^0 = 1$.

(Exponential theorem) Let $a \in \mathbb{R}$.

$$\lim_{x \to a} e^x = e^a, \quad \lim_{x \to a} \sin x = \sin a$$

$\lim_{x \to a} \cos x = \cos a$, $\lim_{x \to a} \sinh x = \sinh a$, $\lim_{x \to a} \cosh x = \cosh a$

Calculus 2

Example 1.0 If $f(x) = x^2+4$ evaluate $\lim\limits_{x\to 0} f(x)$.

Solution: $\lim\limits_{x\to 0} x^2+4 = 0^2+4$, since

$f(x) = x^2+4$ is continuous at $x=0$.

Example 1.2 If $f(x) = \frac{1}{x^2}$ evaluate $\lim\limits_{x\to 0} \frac{1}{x^2}$.

Solution As N gets larger and larger,

$\frac{1}{10N}$ gets closer and closer to 0

and

$$\frac{1}{\left(\frac{1}{10N}\right)^2} = \frac{1}{\frac{1}{10^{2N}}} = 10^{2N} \text{ gets larger and larger.}$$

So $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$, or does not exist.

Example 1.3 If $f(x) = \begin{cases} 1, & \text{if } x<0, \\ 2, & \text{if } x\geq 0 \end{cases}$ then

evaluate $\lim\limits_{x\to 0} f(x)$.

Solution If $x = 10^{-N}$ then $f(x) = 2$

If $x = -10^{-N}$ then $f(x) = 1$

So $\lim\limits_{x\to 0} f(x)$ does not exist.

Example 1.4 If $f(x) = \begin{cases} 2x, & \text{if } x \neq 1 \\ 4, & \text{if } x = 1 \end{cases}$ then

evaluate $\lim_{x \to 1} f(x)$.

Solution: If $x = 1 + 10^{-N}$ then $f(x) = 2(1+10^{-N})$

$= 2 + 2 \cdot 10^{-N}$

If $x = 1 - 10^{-N}$ then $f(x) = 2(1-10^{-N}) = 2 - 2 \cdot 10^{-N}$

So $\lim_{x \to 1} f(x) = 2$.

Example 1.4 Evaluate $\lim_{x \to 2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution: $\lim_{x \to 2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{2^3 + 2 \cdot 2^2 - 1}{5 - 3 \cdot 2} = \frac{8 + 8 - 1}{5 - 6}$

$= \frac{15}{-1} = -15.$

Example 1.5 If $f(x) = e^{-x}$ then evaluate $\lim_{x \to \infty} f(x)$.

Solution: $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x}$

If $N \in \mathbb{Z}$ is very large then $\frac{1}{e^N} < \frac{1}{10^{N/3}} = 10^{-N/3}$

So $\lim_{x \to \infty} e^{-x} = 0$.