Calculus 2 Lecture 19

09.09.2014
A. Ram

Mathematical Modeling: Population models

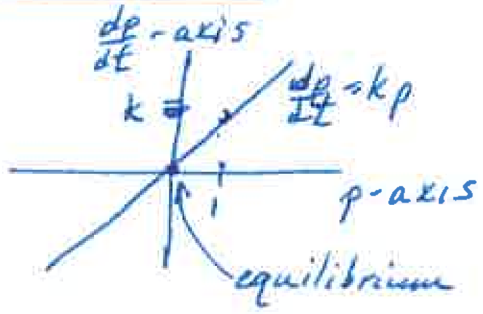
$$\frac{dp}{dt} = kp$$ Mathus Doomsday population model

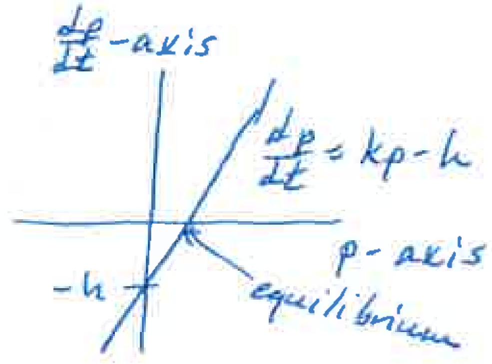
$$\frac{dp}{dt} = kp - h$$ Doomsday model with harvesting

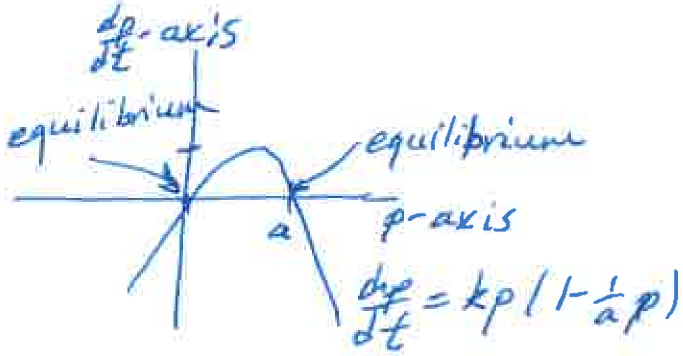
$$\frac{dp}{dt} = kp - \frac{k}{a}p^2$$ logistic model
a = carrying capacity
$-\frac{k}{a}p^2$ is the competition term

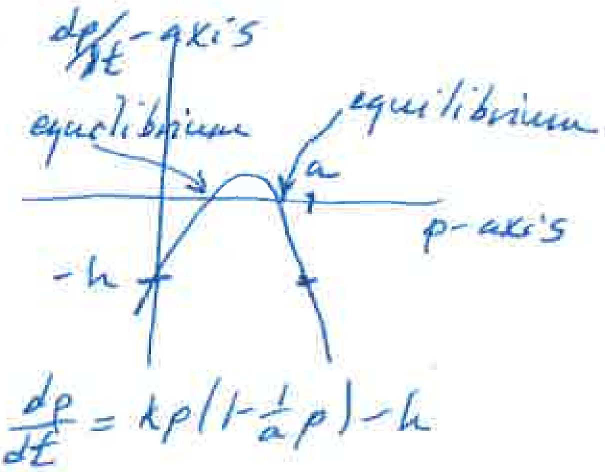
$$\frac{dp}{dt} = kp\left(1-\frac{1}{a}p\right) - h = -\frac{k}{a}p^2 + kp - h$$ logistic model with harvesting

Phase plots









Example 5.12 $\frac{dp}{dt} = 3p-2$.

Solution Since $\frac{1}{3p-2}\frac{dp}{dt} = 1$ then

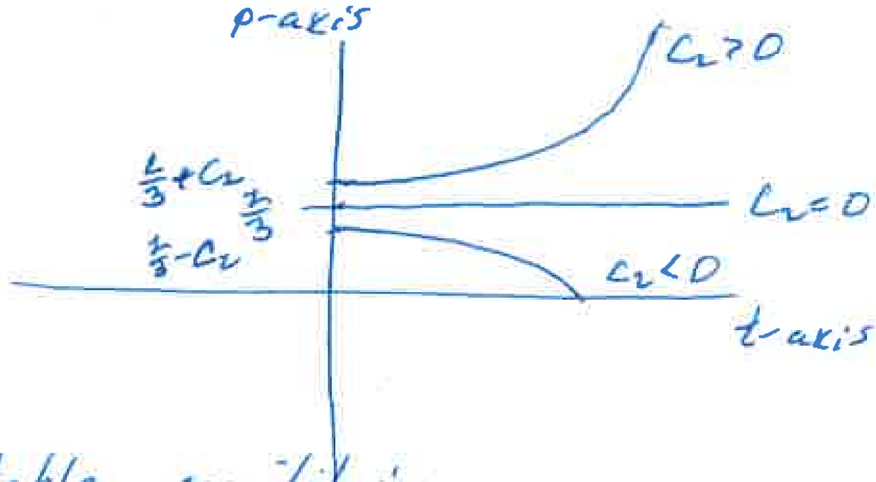
$\int \frac{1}{3p-2}\frac{dp}{dt}\,dt = \int dt$ and $\frac{1}{3}\log(3p-2) = t + c_1$

So $\log(3p-2) = 3t + 3c_1$ and $3p-2 = e^{3t}e^{3c_1} = Ce^{3t}$,

where C is a constant.

So $p = \frac{1}{3}Ce^{3t} + \frac{2}{3} = C_2e^{3t} + \frac{2}{3}$,

where C_2 is a constant



Unstable equilibrium.

~~If $p(0)=1$ then $1 = C_2 + \frac{2}{3}$~~

Example 5.13 $\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right)$

Solution Since $\frac{1}{p(1-\frac{1}{4}p)} \frac{dp}{dt} = 1$ then

$$\int dt = \int \frac{1}{p(1-\frac{1}{4}p)} \frac{dp}{dt}\, dt = \int \frac{4}{p(4-p)}\, dp$$

$$= \int \left(\frac{1}{p} + \frac{1}{4-p}\right) dp.$$

So $\log p - \log(4-p) = t + c_1$

So $\log\left(\frac{p}{4-p}\right) = t + c_1.$

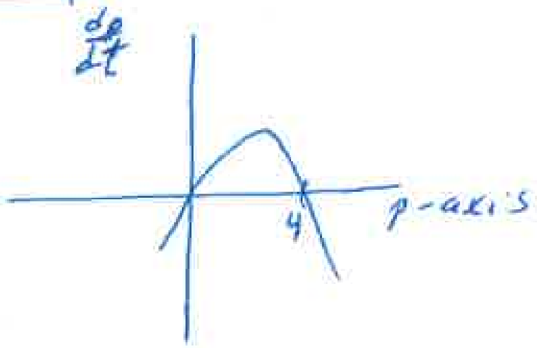
So $\frac{p}{4-p} = Ce^t$ and $\frac{-(4-p)+4}{4-p} = Ce^t$

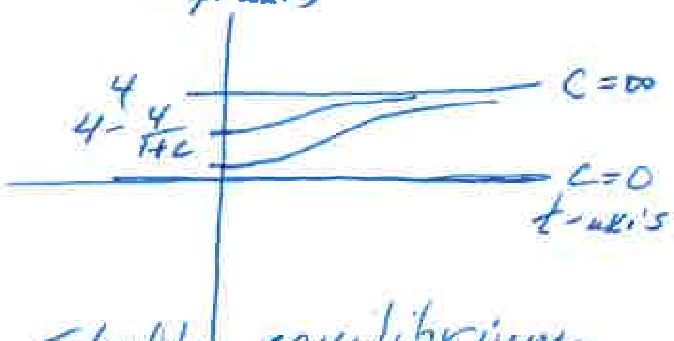
So $\frac{4}{4-p} - 1 = Ce^t$ and $\frac{4}{4-p} = Ce^t + 1.$

So $\frac{4}{Ce^t+1} = 4 - p$ and $p = 4 - \frac{4}{Ce^t+1}$

So $p = \frac{4Ce^t + 4 - 4}{Ce^t+1} = \frac{4Ce^t}{Ce^t+1} = \frac{4C}{C+e^{-t}} = \frac{4}{1+\frac{1}{C}e^{-t}}$

Phase plot **Solution graph**





stable equilibrium.

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Example 5.14 $\frac{dp}{dt} = p\left(1-\frac{p}{4}\right) - \frac{3}{4}$

Solution Since $\frac{1}{p(1-\frac{p}{4})-\frac{3}{4}} \frac{dp}{dt} = 1$ and

$$\frac{1}{p(1-\frac{p}{4})-\frac{3}{4}} = \frac{1}{-\frac{1}{4}(p^2-4p+3)} = \frac{-4}{(p-3)(p-1)}$$

$$= \frac{-2}{p-3} + \frac{2}{p-1} \quad \text{then}$$

$$\int dt = \int \left(\frac{1}{p(1-\frac{p}{4})-\frac{3}{4}}\right)\frac{dp}{dt}\, dt = \int\left(\frac{-2}{p-3}+\frac{2}{p-1}\right) dp$$

and

$$t + c_1 = -2\log(p-3) + 2\log(p-1) = \log\left(\left(\frac{p-1}{p-3}\right)^2\right)$$

So $\left(\frac{p-1}{p-3}\right)^2 = Ce^t$, where C is a constant.

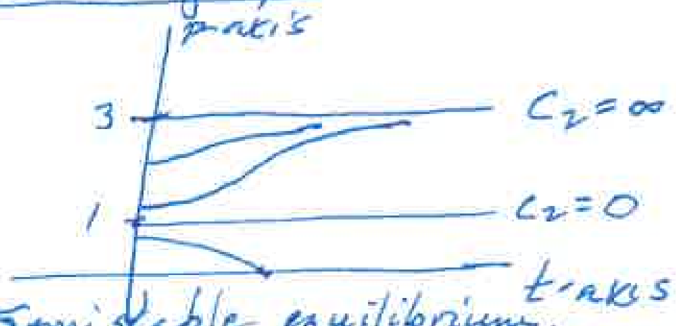
So $\frac{p-1}{p-3} = \pm C_2 e^{\frac{1}{2}t}$ and $p-1 = C_2 e^{\frac{1}{2}t} p - 3C_2 e^{\frac{1}{2}t}$.

So $(1 - C_2 e^{\frac{1}{2}t})p = -3C_2 e^{\frac{1}{2}t}$.

So $p = \frac{3C_2 e^{\frac{1}{2}t}}{C_2 e^{\frac{1}{2}t}-1} = \frac{3}{1 \mp \frac{1}{C_2} e^{-\frac{1}{2}t}}$

Phase plot

Solution graph



Semistable equilibrium.

Example 5.13

If $p(0)=1$ then $1 = \dfrac{4}{1+\frac{1}{c}e^{-0}} = \dfrac{4}{1+\frac{1}{c}}$

and $1+\frac{1}{c}=4$ and $\frac{1}{c}=3$ so that $c=\frac{1}{3}$.

Example 5.14 Find the time until the population dies out if $p(0)=\frac{1}{2}$

Solution

$$\left(\frac{p-1}{p-3}\right)^2 = Ce^t$$

If $p(0)=\frac{1}{2}$ then $\left(\dfrac{\frac{1}{2}-1}{\frac{1}{2}-3}\right)^2 = Ce^0$.

So $C = \left(\dfrac{-\frac{1}{2}}{\frac{5}{2}}\right)^2 = \left(\frac{1}{5}\right)^2 = \frac{1}{25}$

The population dies out when $p=0$. So

$\left(\dfrac{0-1}{0-3}\right)^2 = \frac{1}{25}e^t$. So $\frac{1}{9} = \frac{1}{25}e^t$

So $e^t = \frac{25}{9}$ and $t = \log\left(\frac{25}{9}\right)$