Calculus 2 Lecture 14 Partial Fractions 28 August 2019 A. Ram (1)

Example 4.8 Evaluate $\int \sinh^5 x \cosh^6 x \, dx$

Solution: $\int \sinh^5 x \cosh^6 x \, dx = \int \sinh x (\sinh^2 x)^2 \cosh^6 x \, dx$

$= \int \sinh x (\cosh^2 x - 1)^2 \cosh^6 x \, dx$

$= \int \sinh x (\cosh^4 x - 2\cosh^2 x + 1) \cosh^6 x \, dx$

$= \int \sinh x (\cosh^{10} x - 2\cosh^8 x + \cosh^6 x) \, dx$

$= \frac{1}{11}\cosh^{11} x - \frac{2}{9}\cosh^9 x + \frac{1}{7}\cosh^7 x + c,$

where c is a constant.

Example 4.9 Evaluate $\int \sinh^5 x \cosh^7 x \, dx$.

Solution: $\int \sinh^5 x \cosh^7 x \, dx = \int \sinh^5 x (\cosh^2 x)^3 \cosh x \, dx$

$= \int \sinh^5 x (\sinh^2 x + 1)^3 \cosh x \, dx$

$= \int \sinh^5 x (\sinh^6 x + 3\sinh^4 x + 3\sinh^2 x + 1) \cosh x \, dx$

$= \int (\sinh^{11} x + 3\sinh^9 x + 3\sinh^7 x + \sinh^5 x) \cosh x \, dx$

$= \frac{1}{12}\sinh^{12} x + \frac{3}{10}\sinh^{10} x + \frac{3}{8}\sinh^8 x + \frac{1}{6}\sinh^6 x + c,$

where c is a constant.

Example 4.10 Evaluate $\int \frac{4}{x^2(x+2)}\,dx$.

Solution: Find A, B, C so that

$$\frac{4}{x^2(x+2)} = \frac{Ax+B}{x^2} + \frac{C}{x+2}.$$ Then

$$\frac{4}{x^2(x+2)} = \frac{Ax^2+2Ax+Bx+2B+Cx^2}{x^2(x+2)}$$ so that

$A+C=0$
$2A+B=0$
$2B=4.$

so $B=2$, $A=-1$, $C=1$ and $\frac{4}{x^2(x+2)} = \frac{-x+2}{x^2} + \frac{1}{x+2}$

Then

$$\int \frac{4}{x^2(x+2)}\,dx = \int\left(\frac{-x+2}{x^2} + \frac{1}{x+2}\right)dx$$

$$= \int\left(\frac{-1}{x} + 2x^{-2} + \frac{1}{x+2}\right)dx$$

$$= -\log x - 2x^{-1} + \log(x+2) + c$$

$$= -\frac{2}{x} - \log x + \log(x+2) + c$$

$$= -\frac{2}{x} + \log\left(\frac{x+2}{x}\right) + c = -\frac{2}{x} + \log\left(1+\frac{2}{x}\right) + c,$$

where c is a constant.

Example 4.11 Evaluate $\int \frac{4x}{(x^2+4)(x-2)}\,dx$

Solution Find A, B, C so that

$$\frac{4x}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2} = \frac{\left(Ax^2-2Ax+Bx-2B + Cx^2+4C\right)}{(x^2+4)(x-2)}.$$

So
$$\begin{aligned} A+C&=0 \\ -2A+B&=4 \\ -2B+4C&=0. \end{aligned}$$
So
$$\begin{aligned} A&=-C, \\ 2C+B&=4 \\ -2B+4C&=0. \end{aligned}$$
So
$$\begin{aligned} A&=-C \\ B&=4-2C \\ -2(4-2C)+4C&=0 \end{aligned}$$

So
$$\begin{aligned} A&=-C, \\ B&=4-2C, \\ 8C&=8. \end{aligned}$$
So
$$\begin{aligned} A&=-1 \\ B&=4-2 \\ C&=1 \end{aligned}$$
and
$$\frac{-x+2}{x^2+4} + \frac{1}{x-2} = \frac{4x}{(x^2+4)(x-2)}$$

Then

$$\int \frac{4x}{(x^2+4)(x-2)}\,dx = \int \left(\frac{-x+2}{x^2+4} + \frac{1}{x-2}\right) dx$$

$$= \int \left(\frac{-\frac{1}{2}(2x)}{x^2+4} + \frac{2}{4}\cdot\frac{1}{\left(\frac{x^2}{4}+1\right)} + \frac{1}{x-2}\right) dx$$

$$= -\frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \frac{1}{2}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx + \int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{2}\log(x^2+4) + \arctan\left(\frac{x}{2}\right) + \log(x-2) + c,$$

where c is a constant. //

Example 4.13 Find $\displaystyle\int \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2}\,dx$

Solution:

$$\frac{5x^4+13x^3+6x^2+4}{x^3+2x^2} = \frac{(x^3+2x^2)(5x+3)+4}{x^3+2x^2}$$

$$= 5x+3+\frac{4}{x^3+2x^2} = 5x+3+\frac{4}{x^2+2x^2}$$

$$= 5x+3+\frac{-x+2}{x^2}+\frac{1}{x+2} = 5x+3-\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x+2}$$

Then

$$\int \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2}\,dx = \int \left(5x+3-\frac{1}{x}+2x^{-2}+\frac{1}{x+2}\right)dx$$

$$= \frac{5}{2}x^2+3x-\log x-2x^{-1}+\log(x+2)+c$$

$$= \frac{5}{2}x^2+3x-\frac{2}{x}+\log\left(\frac{x+2}{x}\right)+c$$

$$= \frac{5}{2}x^2+3x-2x^{-1}+\log\left(1+\frac{2}{x}\right)+c,$$

where c is a constant.