Calculus 2 Problem sheet 1

(1.3 a) Evaluate $\lim_{x\to 2}(x^2-3x+5)$

$\lim_{x\to 2}(x^2-3x+5) = 2^2-3\cdot 2+5 = 4-6+5 = 9-6 = 3,$

since x^2-3x+5 is a polynomial and continuous for $x=2$.

(1.3 b) Evaluate $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2} x+3 = 2+3 = 5,$$

since $x+3$ is a polynomial and is continuous at $x=2$.

(1.3c) Evaluate $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$.

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h}$$

$= \lim_{h\to 0}(2x+h) = 2x+0 = 2x$, which is the derivative of x^2.

(1.3d) Evaluate $\lim_{x\to -4}\frac{x^2+7x+12}{x^2+3x-4}$

$$\lim_{x\to -4}\frac{x^2+7x+12}{x^2+3x-4} = \lim_{x\to -4}\frac{(x+4)(x+3)}{(x+4)(x-1)} = \lim_{x\to -4}\frac{x+3}{x-1}$$

$= \frac{-4+3}{-4-1} = \frac{-1}{-5} = \frac{1}{5}$, since $\frac{x+3}{x-1}$ is continuous at $x=-4$.

(1.3e) Evaluate $\lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7}$

$$\lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7} = \lim_{x \to 7} \frac{(\sqrt{x+2}-3)}{(x-7)} \frac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)}$$

$$= \lim_{x \to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x \to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{7+2}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

(1.3f) Evaluate $\lim_{x \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$\lim_{x \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \lim_{x \to -4} \frac{\frac{x+4}{4x}}{4+x} = \lim_{x \to -4} \frac{1}{4x} = \frac{1}{4(-4)} = \frac{-1}{16}.$$

(1.5a) Evaluate $\lim_{x \to 0} x^2 \cos\left(\frac{2\pi}{x}\right)$ using Sandwich theorem.

Since $-1 \leq \cos\theta \leq 1$ and $x^2 > 0$ then

$$x^2(-1) \leq x^2 \cos\left(\frac{2\pi}{x}\right) \leq x^2 \cdot 1. \quad \text{So}$$

$$\lim_{x \to 0}(-x^2) \leq \lim_{x \to 0} x^2 \cos\left(\frac{2\pi}{x}\right) \leq \lim_{x \to 0} x^2. \quad \text{So}$$

$$0 \leq \lim_{x \to 0} x^2 \cos\left(\frac{2\pi}{x}\right) \leq 0. \text{ So } \lim_{x \to 0} x^2 \cos\left(\frac{2\pi}{x}\right) = 0.$$

(1.5b) Evaluate $\lim_{x \to \infty} e^{-2x} \sin x$ using the Sandwich method.

Since $-1 \leq \sin x \leq 1$ and $e^{-2x} > 0$ then

$e^{-2x}(-1) \le e^{-2x}\sin x \le e^{-2x}\cdot 1$. So

$$\lim_{x\to\infty}(-e^{-2x}) \le \lim_{x\to\infty}(e^{-2x}\sin x) \le \lim_{x\to\infty} e^{-2x}. \text{ So}$$

$$0 \le \lim_{x\to\infty}(e^{-2x}\sin x) \le 0, \text{ since } \lim_{x\to\infty} e^{-2x} = 0.$$

So $\lim_{x\to\infty} e^{-2x}\sin x = 0$.

(1.5c) Evaluate $\lim_{x\to 0} x^3\sin(\frac{1}{x})$ using the Sandwich method.

Since $-1 \le \sin(\frac{1}{x}) \le 1$ then $|x^3\sin(\frac{1}{x})| \le |x^3|$.

So $\lim_{x\to 0}|x^3\sin(\frac{1}{x})| \le \lim_{x\to 0}|x^3|$. So

$\lim_{x\to 0}|x^3\sin(\frac{1}{x})| \le 0$. Since $|x^3\sin(\frac{1}{x})| \ge 0$ then

$0 \le \lim_{x\to 0}|x^3\sin(\frac{1}{x})| \le 0$. So $\lim_{x\to 0}|x^3\sin(\frac{1}{x})| = 0$.

So $\lim_{x\to 0} x^3\sin(\frac{1}{x}) = 0$.

(1.22a) Use comparison test to determine

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$$

Since $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ then $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1} \ge \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$.

Since $\frac{1}{2} \le 1$ then $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \infty$ (p-series).

So $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} \geq \infty$. So $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges.

(1.22b) Use comparison test to determine if $\sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n^2+1}$ converges or diverges.

Since $\frac{\sqrt{n}-1}{n^2+1} \leq \frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ then

$0 \leq \sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n^2+1} \leq \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$. Since $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ (a p-series)

converges to a real number L then

$0 \leq \sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n^2+1} \leq L$. So $\sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n^2+1}$ converges.

(1.22c) Use comparison test to determine if $\sum_{n=1}^{\infty} \frac{n}{n^2+n-1}$ converges or diverges.

$\sum_{n=1}^{\infty} \frac{n}{n^2+n-1} = \sum_{n=1}^{\infty} \frac{1}{n+1-\frac{1}{n}}$. Since $\frac{1}{n+1-\frac{1}{n}} \geq \frac{1}{n}$

for $n \geq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n+1-\frac{1}{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n}$. Since

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (to ∞) then $\sum_{n=1}^{\infty} \frac{1}{n+1-\frac{1}{n}} \geq \infty$.

So $\sum_{n=1}^{\infty} \frac{n}{n^2+n-1}$ diverges.

(1.24a) Is $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ convergent or divergent?

Since $\sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2$ then $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent.

(1.24b) Is $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ convergent or divergent?

$\sqrt{\frac{n}{n+1}} = \sqrt{\frac{1}{1+\frac{1}{n}}} > \frac{1}{\sqrt{2}}$. So $\sum_{n=1}^{\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}} = \infty$.

So $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ diverges.

(1.24c) Is $\sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^2}$ convergent or divergent?

Since $0 \le \sin^2 n \le 1$ then $0 \le \frac{\sin^2 n}{1+n^2} \le \frac{1}{1+n^2} \le \frac{1}{n^2}$.

So $\sum_{n=1}^{\infty} 0 \le \sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (p-series).

So $\sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^2}$ converges.

(1.24d) Is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ convergent or divergent?

$\frac{1}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}(\sqrt{n+1})} \ge \frac{1}{\sqrt{n}(\sqrt{n}+\sqrt{n})} = \frac{1}{2n}$. So

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \ge \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (p-series).

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ is divergent.

(1.24e) Is $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ convergent or divergent?

Let us compare n^3 and 2^n:

	1	2	3	4	5	6	7	8	9
n^3	1	8	27	64	125	216	343	512	729
2^n	2	4	8	16	32	128	256	512	1024

As soon as $n \geq 8$ then $2^n \geq n^3$.

So $\sum_{n=1}^{\infty} \frac{n^3}{4^n} = \left(\frac{1}{4} + \frac{8}{16} + \frac{27}{64} + \frac{64}{256} + \cdots + \frac{7^3}{4^7}\right) + \sum_{n=8}^{\infty} \frac{n^3}{4^n}$

$\leq (1+1+1+1+1+1+1) + \sum_{n=8}^{\infty} \frac{2^n}{4^n}$

$\leq 7 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \leq 7 + \frac{1}{1-\frac{1}{2}}$ (geometric series)

$= 7 + 2 = 9.$

So $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ converges.

(1.24f) Is $\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n}$ convergent or divergent?

$\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n} = \sum_{n=0}^{\infty} 3^{-1}\left(\frac{3}{2}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n \geq \frac{1}{3} \sum_{n=0}^{\infty} 1 = \infty.$

So $\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n}$ diverges.

(1.25a) Is the sequence $a_n = \left(\frac{n}{n+7}\right)^n$ convergent?

$\lim_{n\to\infty} \left(\frac{n}{n+7}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{7}{n}\right)^n} = \frac{1}{e^7}$. So $a_n = \left(\frac{n}{n+7}\right)^n$ is convergent.

(1.25b) Is $\sum_{n=1}^{\infty}\left(\frac{n}{n+7}\right)^n$ convergent?

Since $\lim_{n\to\infty}\left(\frac{n}{n+7}\right)^n = \frac{1}{e^7}$ there exists $k \in \mathbb{Z}_{>0}$ such that

if $n \geq k$ then $\left(\frac{n}{n+7}\right)^n \geq \frac{1}{2}\cdot\frac{1}{e^7}$. So

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+7}\right)^n = \sum_{n=1}^{k}\left(\frac{n}{n+7}\right)^n + \sum_{n=k+1}^{\infty}\left(\frac{n}{n+7}\right)^n \geq \sum_{n=1}^{k}\left(\frac{n}{n+7}\right)^n + \sum_{k=1}^{\infty}\frac{1}{2}\frac{1}{e^7}$$

$$\geq \sum_{n=1}^{k} 0 + \frac{1}{2e^7}\sum_{n=k+1}^{\infty} 1 = \frac{1}{2e^7}\cdot\infty.$$ So $\sum_{n=1}^{\infty}\left(\frac{n}{n+7}\right)^n$ diverges.

(1.25c) Is $\sum_{n=1}^{\infty}\frac{9n^4+2n^3+5}{3n^3-n^2}$ convergent?

$$\sum_{n=1}^{\infty}\frac{9n^4+2n^3+5}{3n^5-n^2} = \sum_{n=1}^{\infty}\frac{9+\frac{2}{n}+\frac{5}{n^4}}{3n-\frac{1}{n^3}} \geq \sum_{n=1}^{\infty}\frac{9}{3n-\frac{1}{n^2}} \geq \sum_{n=1}^{\infty}\frac{9}{2n}$$

$$= \frac{9}{2}\sum_{n=1}^{\infty}\frac{1}{n} = \frac{9}{2}\cdot\infty,$$ (harmonic series).

So $\sum_{n=1}^{\infty}\frac{9n^4+2n^3+5}{3n^5-n^2}$ diverges.

(1.25d) Is the sequence $a_n = \frac{9n^4+2n^3+5}{3n^5-n^2}$ convergent?

$$\lim_{n\to\infty}\frac{9n^4+2n^3+5}{3n^5-n^2} = \lim_{n\to\infty}\frac{\frac{9}{n}+\frac{2}{n^2}+\frac{5}{n^5}}{3-\frac{1}{n^3}} = \frac{0+0+0}{3-0} = 0.$$

So $a_n = \frac{9n^4+2n^2+5}{3n^5-n^2}$ is convergent.

(1.25e) Is the sequence $a_n = \frac{\sin(2n)}{n}$ convergent?

Since $|\sin(2n)| \le 1$ then $\lim_{n\to\infty} \left|\frac{\sin(2n)}{n^3}\right| \le \lim_{n\to\infty} \frac{1}{n^3} = 0$

So $\lim_{n\to\infty} \left|\frac{\sin(2n)}{n^3}\right| = 0$. So $\lim_{n\to\infty} \frac{\sin(2n)}{n^3} = 0$.

So $a_n = \frac{\sin(2n)}{n^3}$ is a convergent sequence.

(1.25f) Is $\sum_{n=0}^{\infty} \frac{5^{n+2}}{7^n}$ convergent?

$$\sum_{n=0}^{\infty} \frac{5^{n+2}}{7^n} = \sum_{n=0}^{\infty} 25\left(\frac{5}{7}\right)^n = 25 \sum_{n=0}^{\infty} \left(\frac{5}{7}\right)^n = 25\left(\frac{1}{1-\frac{5}{7}}\right) \quad \text{(geometric series)}$$

$$= 25 \cdot \frac{7}{2}.$$

So $\sum_{n=0}^{\infty} \frac{5^{n+2}}{7^n}$ is convergent.

(1.16a) Using the sandwich method find $\lim_{n\to\infty} \frac{2+(-1)^n}{n^2}$.

Since $-1 \le (-1)^n \le 1$ then $\frac{2-1}{n^2} \le \frac{2+(-1)^n}{n^2} \le \frac{2+1}{n^2}$. So

$$\lim_{n\to\infty} \frac{1}{n^2} \le \lim_{n\to\infty} \frac{2+(-1)^n}{n^2} \le \lim_{n\to\infty} \frac{3}{n^2}.$$

So $0 \le \lim_{n\to\infty} \frac{2+(-1)^n}{n^2} \le 0$. So $\lim_{n\to\infty} \frac{2+(-1)^n}{n^2} = 0$.

(1.16b) Use the Sandwich method to find $\lim_{n\to\infty} (3^n+1)^{1/n}$.

Since $3^n \le 3^n+1 \le 3^{n+1}$ then $(3^n)^{1/n} \le (3^n+1)^{1/n} \le (3^{n+1})^{\frac{1}{n}}$.

So $3 \le (3^n+1)^{1/n} \le 3^{1+\frac{1}{n}}$. So $\lim_{n\to\infty} 3 \le \lim_{n\to\infty} (3^n+1)^{\frac{1}{n}} \le \lim_{n\to\infty} 3^{1+\frac{1}{n}}$.

So $3 \le \lim_{n\to\infty} (3^n+1)^{1/n} \le 3^{1+0}$. So $\lim_{n\to\infty} (3^n+1)^{1/n} = 3$.

(1.16c) Use the Sandwich method to find $\lim_{n\to\infty} \frac{n!}{n^n}$.

Since $\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \cdots \frac{\frac{n}{2}}{n} \cdot \frac{(\frac{n}{2}+1)}{n} \cdot \cdots \frac{n-1}{n} \cdot \frac{n}{n}$

$$\le \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} \cdot 1 \cdots 1 \cdot 1 = \frac{1}{2^{n/2}}.$$

So $0 \le \frac{n!}{n^n} \le \left(\frac{1}{2}\right)^{n/2}$. So $\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n/2}$

So $0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le 0$. So $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.