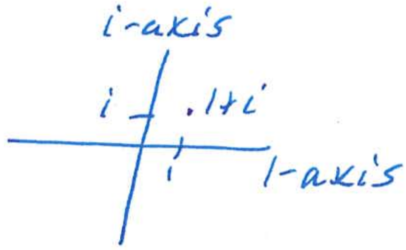
A. Ram

Calculus 1 Lecture 8

Example 1.69 Find $\left(\frac{2}{1+i}\right)^{14}$ in exponential and Cartesian form.

Solution:



$$1+i = \sqrt{2}\, e^{i\pi/4} = 2^{1/2} e^{i\pi/4}$$

So
$$\left(\frac{2}{1+i}\right)^{14} = \left(\frac{2}{2^{1/2} e^{i\pi/4}}\right)^{14} = \left(2^{1/2} e^{-i\pi/4}\right)^{14} = 2^{14/2} e^{-i14/4}$$

$$= 2^7 e^{-i\frac{7}{2}\pi} e^{i2\pi} e^{i2\pi} = 2^7 e^{i\pi(4-\frac{7}{2})} = 2^7 e^{i\pi/2} = 2^7 i$$

$$= 128 i. \ //$$

Example 1.71 Find the set of cube roots of -8.

Solution: $\{z \in \mathbb{C} \mid z^3 = -8\}$

$$= \left\{ re^{i\theta} \in \mathbb{C} \mid (re^{i\theta})^3 = 8e^{i\pi} \text{ or } (re^{i\theta})^3 = 8e^{i3\pi} \text{ or } (re^{i\theta})^3 = 8e^{i5\pi} \right\}$$

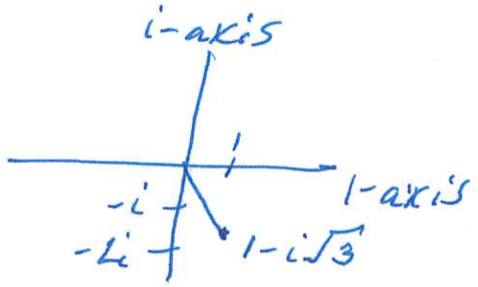
$$= \left\{ re^{i\theta} \in \mathbb{C} \mid re^{i\theta} = 8^{\frac{1}{3}} e^{i\pi/3} \text{ or } re^{i\theta} = 8^{\frac{1}{3}} e^{i3\pi/3} \text{ or } re^{i\theta} = 8^{\frac{1}{3}} e^{i\frac{5\pi}{3}} \right\}$$

$$= \left\{ re^{i\theta} \in \mathbb{C} \mid re^{i\theta} = 2e^{i\pi/3} \text{ or } re^{i\theta} = 2e^{i\pi} \text{ or } re^{i\theta} = 2e^{i5\pi/3} \right\}$$

$$= \left\{ 2e^{i\pi/3}, 2e^{i\pi}, 2e^{i5\pi/3} \right\}$$

Example 1.72 Find the 4^{th} roots of $1-i\sqrt{3}$.

Solution:



$$1-i\sqrt{3} = \sqrt{1+3}\,e^{-i\pi/3} = 2e^{-i\pi/3}$$

If $(re^{i\theta})^4$ is $2e^{-i\pi/3}$ or $2e^{+i5\pi/3}$ or $2e^{11i\pi/3}$ or $2e^{i17\pi/3}$

then $re^{i\theta}$ is $2^{1/4}e^{-i\pi/12}$ or $2^{1/4}e^{i5\pi/12}$ or $2^{1/4}e^{i\pi 11/12}$

or $2^{1/4}e^{i17\pi/12}$

So the 4^{th} roots of $1-i\sqrt{3}$ are

$$2^{1/4}e^{-i\pi/12} = 2^{1/4}e^{+i\pi 23/12} \quad \text{and}$$

$$2^{1/4}e^{i5\pi/12} \quad \text{and} \quad 2^{1/4}e^{i\pi 11/12} \quad \text{and} \quad 2^{1/4}e^{i17\pi/12}$$

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Example 1.73 Prove that $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$

Solution Using $e^{i\theta} = \cos(\theta) + i\sin\theta$

and $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

and $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$,

$$\frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta))$$

$$= \frac{1}{2}(\cos\theta + i\sin\theta + \cos\theta + -i\sin\theta)$$

$$= \frac{1}{2}\,2\cos\theta = \cos\theta.\ //.$$

Example 1.76 Expand $(a+b)^4$ and $(a-b)^4$.

Solution:

$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$

$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2)$

$= a^3+2a^2b+ab^2$

$+ a^2b+2ab^2+b^3$

$= a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = (a+b)(a+b)^3 = (a+b)(a^3+3a^2b+3ab^2+b^3)$

$= a^4+3a^3b+3a^2b^2+ab^3$

$+ a^3b+3a^2b^2+3ab^3+b^4$

$= a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

$(a-b)^2 = (a-b)(a-b) = a^2-ab$
$-ab+b^2 = a^2-2ab+b^2$

$(a-b)^3 = (a-b)(a-b)^2 = (a-b)(a^2-2ab+b^2)$

$= a^3-2a^2b+ab^2$

$-a^2b+2ab^2-b^3 = a^3-3a^2b+3ab^2-b^3$

$(a-b)^4 = (a-b)(a-b)^3 = (a-b)(a^3-3a^2b+3ab^2-b^3)$

$= a^4-3a^3b+3a^2b^2-ab^3$

$-a^3b+3a^2b^2-3ab^3+b^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

<u>Example 1.77</u> Express $\sin^4\theta$ as a sum of sines or cosines of multiples of θ

<u>Solution</u> $\sin^4\theta = \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^4$

$$= \frac{1}{2^4 i^4}\left(e^{i\theta} - e^{-i\theta}\right)^4$$

$$= \frac{1}{2^4}\Big((e^{i\theta})^4 - 4(e^{i\theta})^3 e^{-i\theta} + 6(e^{i\theta})^2(e^{-i\theta})^2 - 4e^{i\theta}(e^{-i\theta})^3 + (e^{-i\theta})^4\Big)$$

$$= \frac{1}{2^4}\Big(e^{i4\theta} - 4e^{i3\theta}e^{i(-\theta)} + 6e^{i2\theta}e^{-i2\theta} - 4e^{i\theta}e^{-i3\theta} + e^{-i4\theta}\Big)$$

$$= \frac{1}{2^4}\left(e^{i4\theta} - 4e^{i2\theta} + 6 - 4e^{-i2\theta} + e^{-i4\theta}\right)$$

$$= \frac{1}{2^4}\left(e^{i4\theta} + e^{-i4\theta} - 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6\right)$$

$$= \frac{1}{2^4}\left(2\cos(4\theta) - 4\cdot 2\cos(2\theta) + 6\right)$$

$$= \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8} \quad //$$

<u>Example 1.78</u> Use the quadratic formula to factorise $p(z) = z^2 - 2iz + 2$.

<u>Solution</u> By the quadratic formula $z^2 - 2iz + 2 = 0$ when

$$z = \frac{-(-2i) \pm \sqrt{(2i)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$= \frac{2i \pm \sqrt{-4-8}}{2} = i \pm \sqrt{-1-2} = i \pm \sqrt{-3}$$

$$= i \pm \sqrt{-1}\sqrt{3} = i \pm \sqrt{3}i = (1+\sqrt{3})i \text{ or } (1-\sqrt{3})i.$$

So

$$z^2 - 2iz + 2 = (z - (1+\sqrt{3})i)(z - (1-\sqrt{3})i)$$

<u>Example 1.81</u> Let $P(z) = z^3 - 3iz^2 - 2z$. Factor $P(z)$, sketch the roots of $P(z)$ and determine if the nonreal roots come in conjugate pairs.

<u>Solution:</u> $P(z) = z^3 - 3iz^2 - 2z$

$$= z(z^2 - 3iz - 2)$$

$$= z(z - \sqrt{5} - \tfrac{3}{2}i)(z + 5 - \tfrac{3}{2}i).$$

since, by the quadratic formula, $z^2 - 3iz - 2 = 0$

when $z = \dfrac{3i \pm \sqrt{(3i)^2 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{3}{2}i \pm \sqrt{-3+8} = \sqrt{5} + \frac{3}{2}i$

or $-\sqrt{5} + \frac{3}{2}i$

The roots of $P(z)$ are

$$0, \ \sqrt{5} + \tfrac{3}{2}i, \ -\sqrt{5} + \tfrac{3}{2}i$$

Then $0 \in \mathbb{R}$ and $\overline{\sqrt{5} + \frac{3}{2}i} = \sqrt{5} - \frac{3}{2}i \neq -\sqrt{5} + \frac{3}{2}i$

So the nonreal roots do not ~~come~~ form a conjugate pair.