Calculus I Lect. 7

13.07.2019 ①
B. Rem

Example 1.59 Evaluate $\left|\dfrac{-2(3-i)(5+2i)}{(1+3i)(7-i)}\right|$.

Solution:

Using that $|zw| = |z||w|$ and $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$,

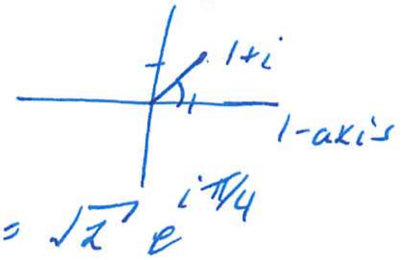
$$\left|\frac{-2(3-i)(5+2i)}{(1+3i)(7-i)}\right| = \frac{|-2|\,|3-i|\,|5+2i|}{|1+3i|\cdot|7-i|}$$

$$= \frac{\sqrt{2^2}\sqrt{3^2+(-1)^2}\sqrt{5^2+2^2}}{\sqrt{1^2+3^2}\sqrt{7^2+(-1)^2}} = \frac{2\sqrt{4}\sqrt{29}}{\sqrt{4}\sqrt{50}}$$

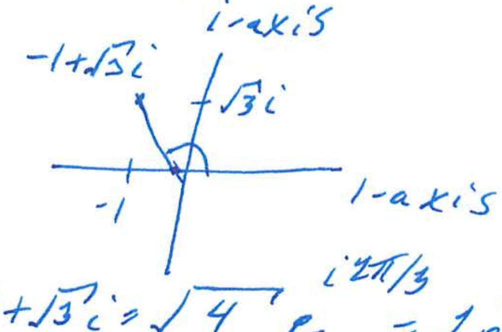
$$= \frac{2\sqrt{29}}{5\sqrt{2}} = \frac{\sqrt{2}\sqrt{29}}{5} = \frac{\sqrt{58}}{5}. //$$

Example 1.60 Let $z_1 = (1+i)(-1+\sqrt{3})$ and $z_2 = \dfrac{-i}{-2+2i}$. Find $\operatorname{Arg}(z_1)$ and $\operatorname{Arg}(z_2)$.

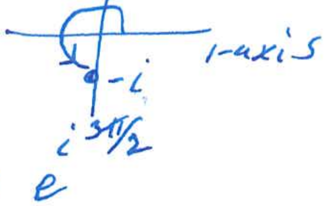
Solution



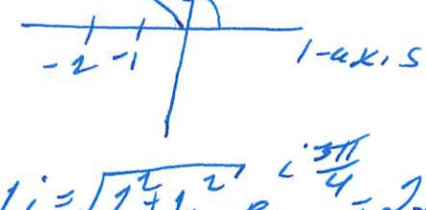
$1+i = \sqrt{2}\, e^{i\pi/4}$



$-1+\sqrt{3}i = \sqrt{4}\, e^{i2\pi/3} = 2e^{i2\pi/3}$



$-i = e^{i3\pi/2}$



$-2+2i = \sqrt{2^2+2^2}\, e^{i\frac{3\pi}{4}} = 2\sqrt{2}\, e^{i3\pi/4}$

So $z_1 = (1+i)(-1+\sqrt{3}) = \sqrt{2}\, e^{i\pi/4} \cdot 2 e^{i 2\pi/3} = 2\sqrt{2}\, e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)}$

So $\mathrm{Arg}(z_1) = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{3\pi + 8\pi}{4 \cdot 3} = \left(\frac{3}{12}+\frac{8}{12}\right)\pi = \pi \cdot \frac{11}{12} = \frac{11}{12}\pi.$

and

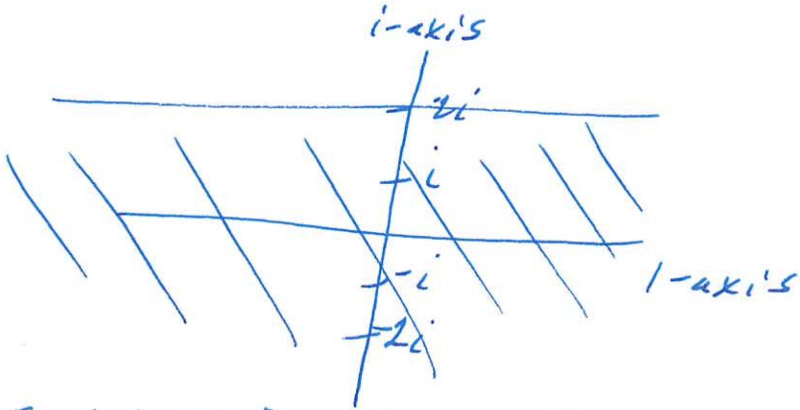
$$z_2 = \frac{-i}{-2+2i} = \frac{e^{i 3\pi/2}}{2\sqrt{2}\, e^{i 3\pi/4}} = \frac{1}{2\sqrt{2}} e^{i\left(\frac{3\pi}{2}-\frac{3\pi}{4}\right)}$$

$$= \frac{1}{2\sqrt{2}} e^{i\frac{3}{2}\pi\left(1-\frac{1}{2}\right)} = \frac{1}{2\sqrt{2}} e^{i\frac{3}{4}\pi}$$

So $\mathrm{Arg}(z_2) = \frac{3}{4}\pi.$ //

Example 1.6 / Sketch $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 2\}$

Solution



$\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 2\} = \{x+yi \mid y < 2\}$

Example 1.62 Sketch $C=\{z\in\mathbb{C} \mid |z-i|=2\}$

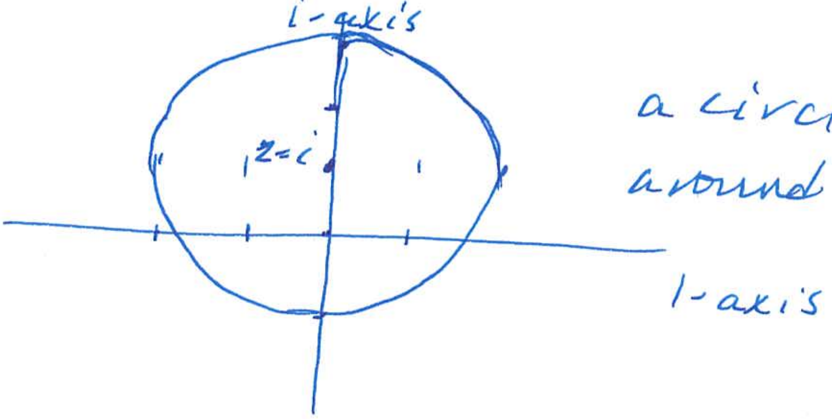
Solution $C=\{z\in\mathbb{C} \mid |z-i|=2\}$

$=\{x+iy \mid |x+iy-i|=2\}$

$=\{x+iy \mid |x+(y-1)i|=2\}$

$=\{x+iy \mid \sqrt{x^2+(y-1)^2}=2\}$

$=\{x+iy \mid x^2+(y-1)^2=4\}.$

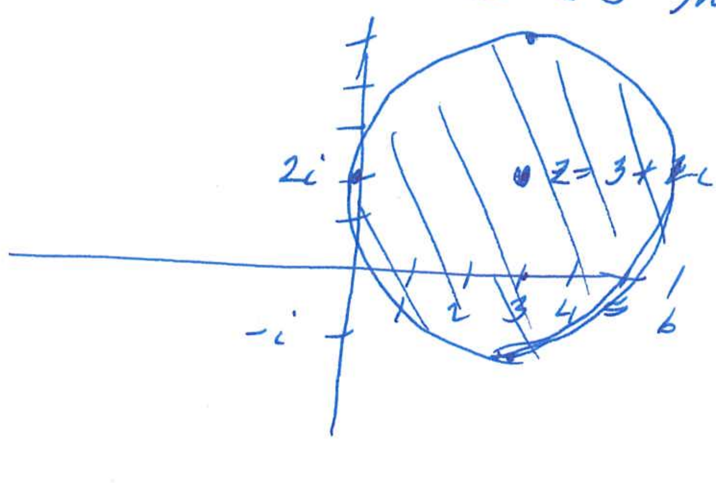


a circle of radius 2 around $z=i$.

Example 1.63 Sketch $\{z\in\mathbb{C} \mid |z-3-2i|\le 3\}$

Solution $\{z\in\mathbb{C} \mid |z-3-2i|\le 3\}=\{z\in\mathbb{C} \mid |z-(3+2i)|\le 3\}$

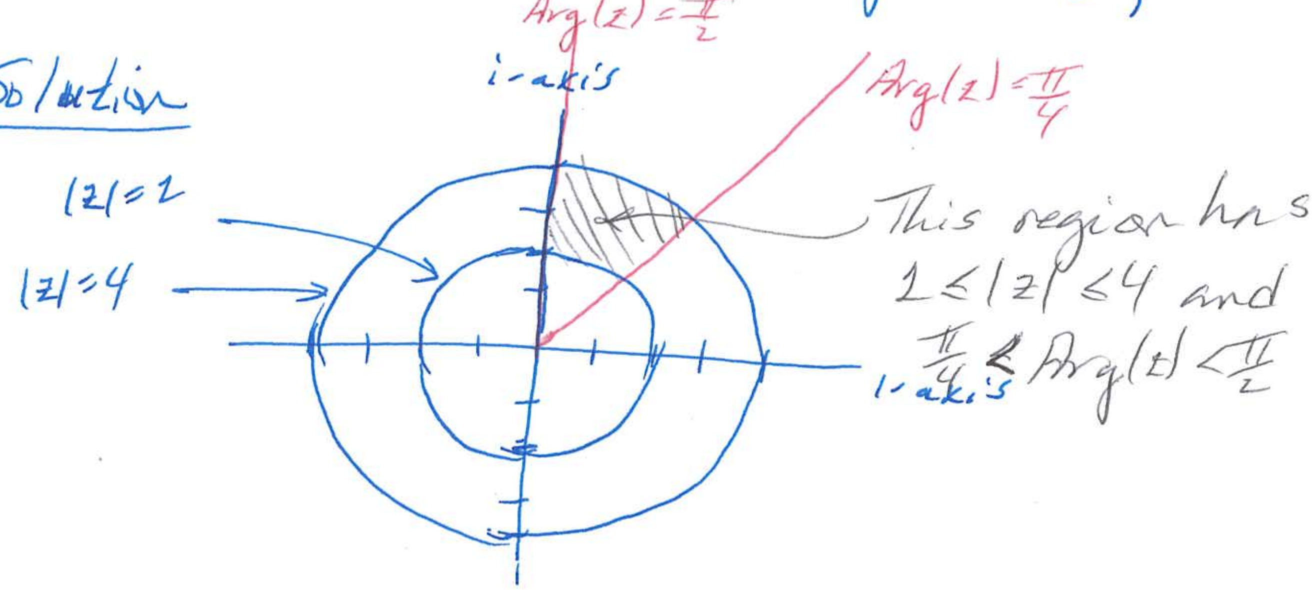
$=\{$points $z\in\mathbb{C}$ of distance ≤ 3 from $3+2i\}$.



Example 1.64 Sketch

$$\{z \in \mathbb{C} \mid 2 \le |z| \le 4 \text{ and } \tfrac{\pi}{4} < \mathrm{Arg}(z) < \tfrac{\pi}{2}\}$$

Solution



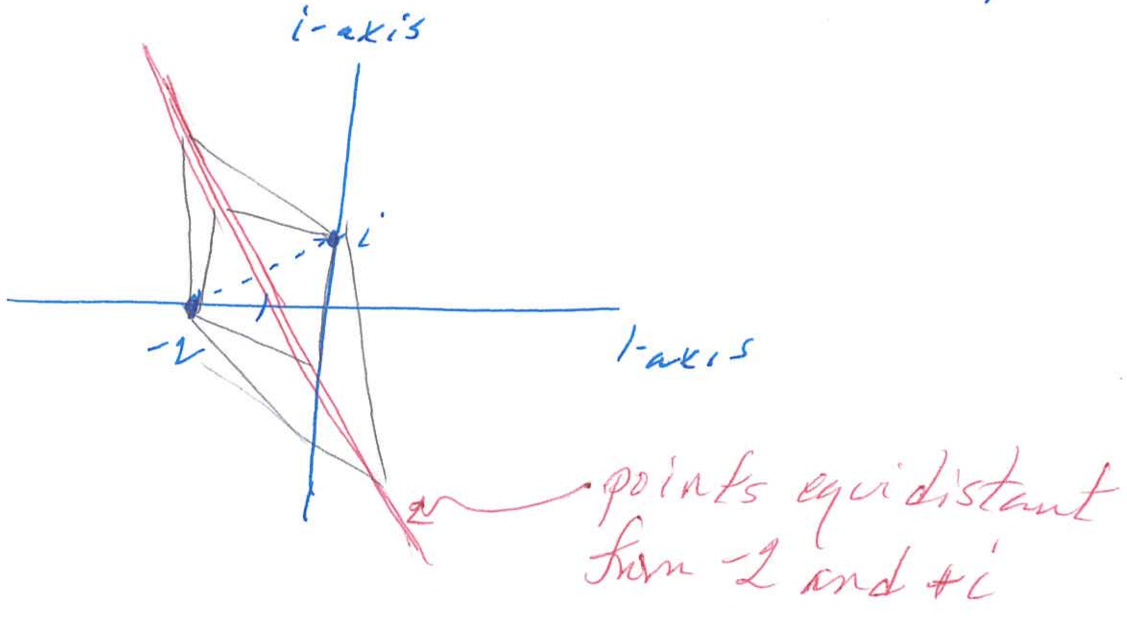
Example 1.65 Sketch

$$\{z \in \mathbb{C} \mid |z+2| = |z+i|\}$$

Solution

$$\{z \in \mathbb{C} \mid |z+2| = |z+i|\} = \left\{z \in \mathbb{C} \;\middle|\; \begin{array}{l}\text{distance } z \text{ to } -2 \\ \text{equals} \\ \text{distance } z \text{ to } -i\end{array}\right\}$$

$= \{$points in $\mathbb{C}$ equidistant from -2 and $-i\}$



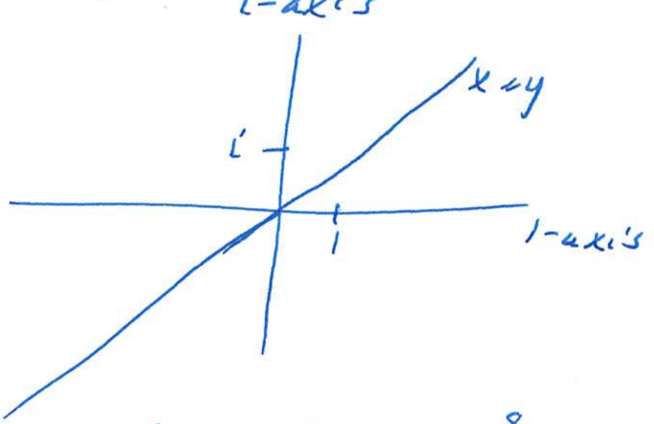
Example 1.66 Sketch $\{z \in \mathbb{C} \mid z = i\bar{z}\}$.

Solution: $\{z \in \mathbb{C} \mid z = i\bar{z}\} = \{x+iy \mid x+iy = i\overline{(x+iy)}\}$

$= \{x+iy \mid x+iy = i(x-iy)\}$

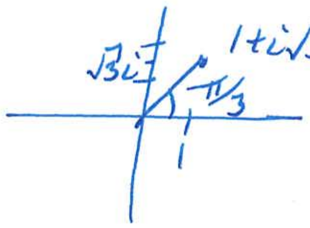
$= \{x+iy \mid x+iy = ix+y\} = \{x+iy \mid x+iy = y+ix\}$

$= \{x+iy \mid x=y\}$.



Example 1.68 Find $(1+i\sqrt{3})^8$

Solution



$1+i\sqrt{3} = \sqrt{1+3}\, e^{i\pi/3}$

$= 2e^{i\pi/3}$

So $(1+i\sqrt{3})^8 = (2e^{i\pi/3})^8 = 2^8 e^{i8\pi/3} = 256\, e^{i\left(\frac{6\pi}{3}+\frac{2\pi}{3}\right)}$

$= 256\, e^{i\left(2\pi+\frac{2\pi}{3}\right)} = 256\, e^{i\frac{2\pi}{3}}$

$= 256\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 256\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 128(-1+\sqrt{3}i)$

$= -128 + 128\sqrt{3}i$.