MAST 10005 Slide
A. Ram 23.07.2019

Complex numbers

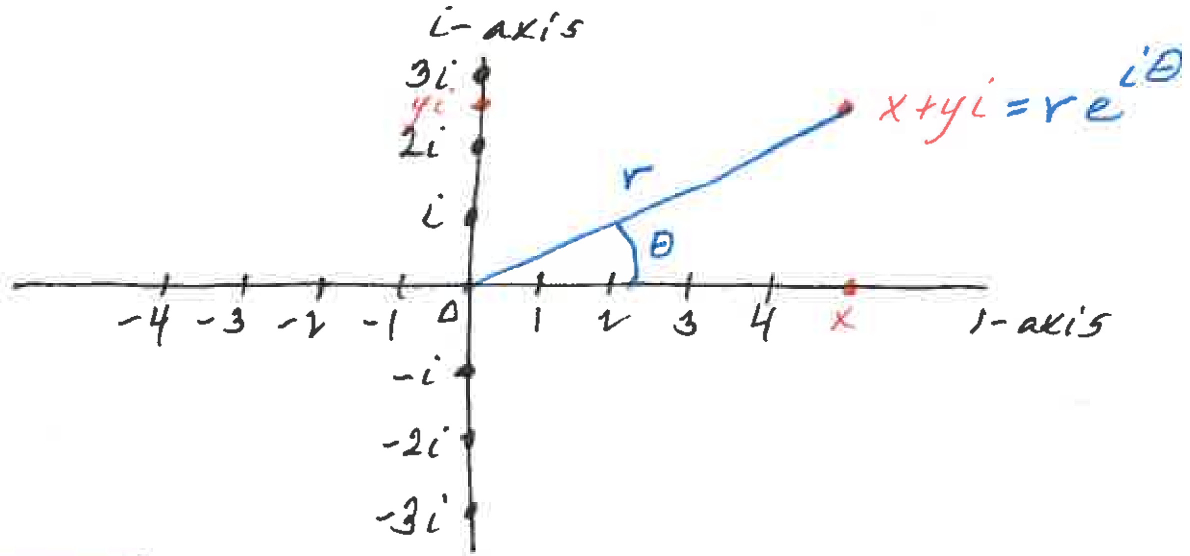
$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$ with $i^2 = -1$.

$(a+bi)+(c+di) = (a+c)+(b+d)i$ (addition)

$c(a+bi) = ca + cbi$ (scalar multiplication)

$(a+bi)(c+di) = ac+adi+bci+bdi^2$

$= (ac-bd)+(ad+bc)i$ (multiplication)



Graphing of real and complex numbers.

$\overline{a+bi} = a-bi$ (conjugation)

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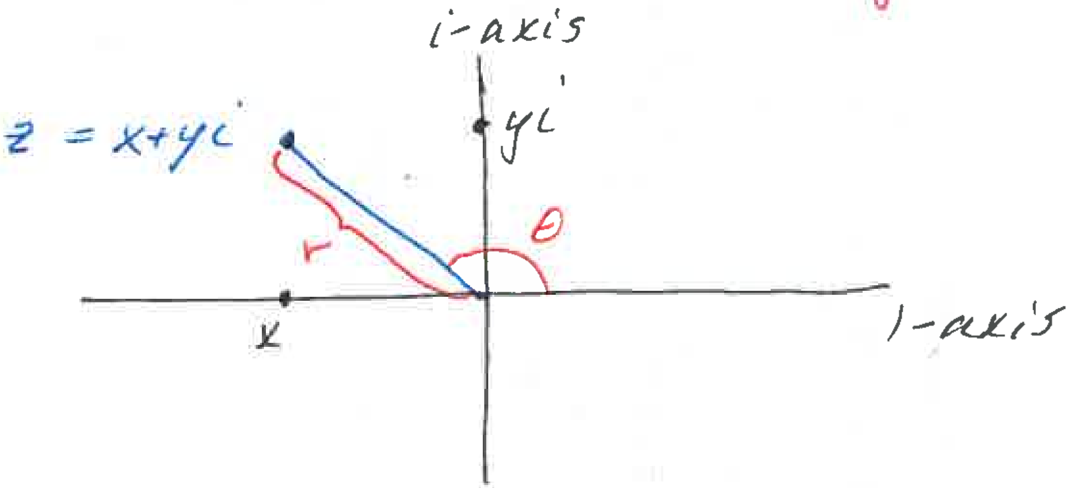
Complex numbers

$z = \underbrace{x}_{\mathrm{Re}(z)} + \underbrace{y}_{\mathrm{Im}(z)} i = \underbrace{r}_{|z|} e^{i\overbrace{\theta}^{\arg(z)}}$ with $x \in \mathbb{R}$, $y \in \mathbb{R}$

$r \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}_{[0, 2\pi)}$



$|z|$ is the <u>modulus</u> of z.

$re^{i\theta} = r\cos\theta + r\sin\theta\, i$, since $x = r\cos\theta$, $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$

Example 1.38 Graph solutions of 6, 7 August 2019
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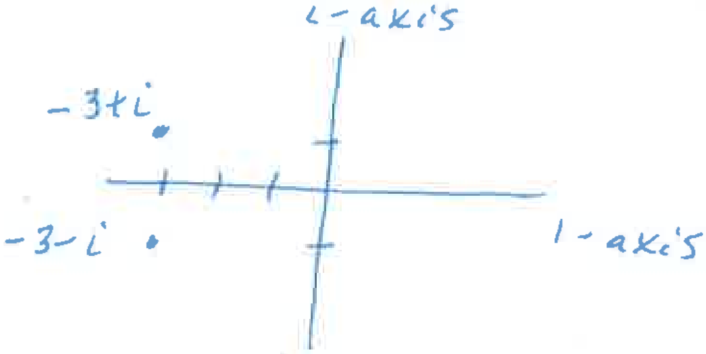
$z^2 - 6z + 10 = 0$.

Solution $0 = z^2 - 6z + 10 = z^2 - 6z + 9 + 1$

$= (z-3)^2 + 1 = (z-3+i)(z-3-i)$

So $z = -3+i$ or $z = -3-i = \overline{-3+i}$.



Example 1.40 Let $z = x+iy$ and $w = a+ib$.

(a) Prove that $z + \bar{z} = 2x = 2\,\mathrm{Re}(z)$

(b) Prove that $z - \bar{z} = 2yi = 2\,\mathrm{Im}(z)\,i$

(c) Prove that $z\bar{z} = x^2 + y^2$

Solution: (a) $z + \bar{z} = (x+iy) + (x-iy) = 2x = 2\,\mathrm{Re}(z)$

(b) $z - \bar{z} = (x+iy) - (x-iy) = 2iy = 2yi = 2\,\mathrm{Im}(z)\,i$.

(c) $z\bar{z} = (x+iy)(x-iy) = x^2 - xiy + iyx - i^2y^2$

$= x^2 - (-1)y^2 = x^2 + y^2$.

Example 1.41 Simplify $z = \dfrac{1+2i}{-1+3i}$.

Solution:
$$z = \frac{1+2i}{-1+3i} = \frac{(1+2i)(-1-3i)}{(-1+3i)(-1-3i)}$$
$$= \frac{-1-3i-2i-6i^2}{1+3^2} = \frac{-1+7-5i}{10}$$
$$= +\frac{6}{10} - \frac{1}{2}i = \frac{3}{5} - \frac{1}{2}i.$$

Example 1.42 Find $\mathrm{Re}\left(\dfrac{1+5i}{2-2i}\right)$ and $\mathrm{Im}\left(\dfrac{1+5i}{2-2i}\right)$.

Solution:
$$\frac{1+5i}{2-2i} = \frac{(1+5i)(2+2i)}{(2-2i)(2+2i)} = \frac{2+12i-10}{4+4}$$
$$= \frac{-8}{8} + \frac{12}{8}i = -1 + \frac{3}{4}i.$$

So
$$\mathrm{Re}\left(\frac{1+5i}{2-2i}\right) = -1 \quad \text{and} \quad \mathrm{Im}\left(\frac{1+5i}{2-2i}\right) = \frac{3}{4}.$$

Example 1.43 Calculate $(1+\sqrt{3}i)^6$.

Solution 1:
$$(1+\sqrt{3}i)^6 = \left((1+\sqrt{3}i)(1+\sqrt{3}i)\right)^3$$
$$= (1+2\sqrt{3}i-3)^3 = (-2+2\sqrt{3}i)^3 = (-2+2\sqrt{3}i)^2(-2+2\sqrt{3}i)$$
$$= (4-2\cdot 4\sqrt{3}i - 4\cdot 3)(-2+2\sqrt{3}i)$$
$$= (-8-8\sqrt{3}i)(-2+2\sqrt{3}i) = 16 + 0i \cancel{-4\sqrt{3}i} + 48 = 64.$$

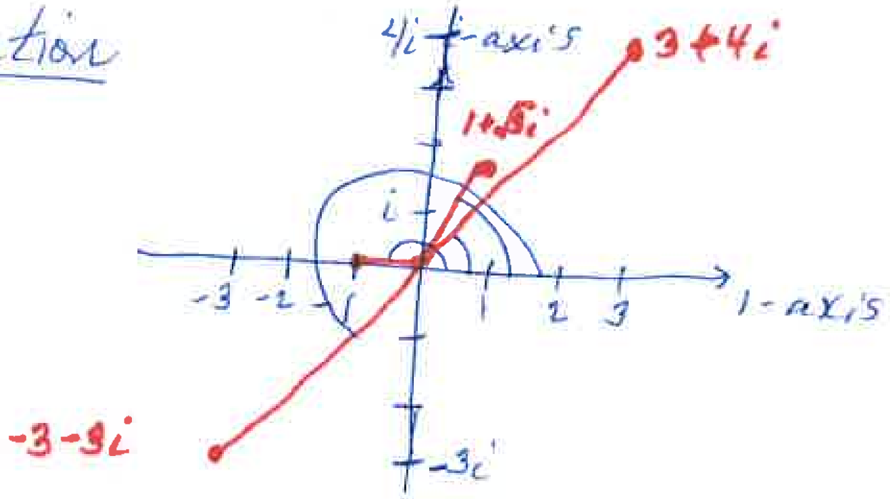
<u>Solution 2:</u> $(1+\sqrt{3}i)^6 = \left(2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)^6 = 2^6\left(e^{i\pi/3}\right)^6$

$= 64\left(e^{i\frac{6\pi}{3}}\right) = 64\left(e^{i2\pi}\right) = 64.$

<u>Example 1.46</u> Find the modulus and argument of

$1+\sqrt{3}i, \quad -3-3i, \quad 3+4i, \quad -1$

<u>Solution</u>



$|1+\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\operatorname{Arg}(1+\sqrt{3}i) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}$

$|-3-3i| = \sqrt{(-3)^2+(-3)^2} = \sqrt{9+9} = 3\sqrt{2}$

$\operatorname{Arg}(-3-3i) = \arctan\left(\frac{-3}{-3}\right) = -\frac{\pi}{2}-\frac{\pi}{4} = -\frac{3\pi}{4}$

$|3+4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

$\operatorname{Arg}(3+4i) = \arctan\left(\frac{4}{3}\right) = \arctan\left(\frac{4}{3}\right).$

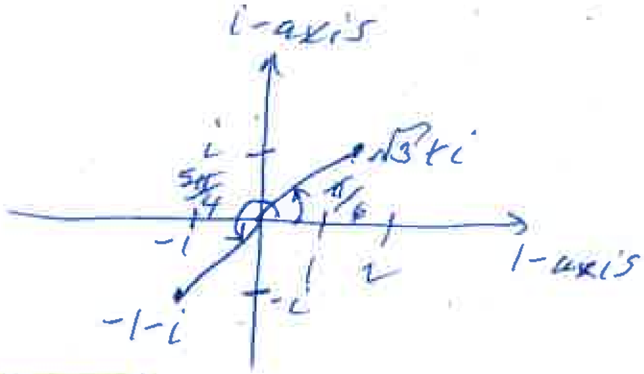
$|-1| = 1$

$\operatorname{Arg}(-1) = \arctan\left(\frac{0}{-1}\right) = \pi.$

Example 1.48 Express the following in polar form:

$$z = \sqrt{3} + i \quad \text{and} \quad z = -1 - i$$

Solution



$$|\sqrt{3}+i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2 \quad \text{and}$$

$$\mathrm{Arg}(\sqrt{3}+i) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \quad \text{so that}$$

$$\sqrt{3} + i = 2e^{i\pi/6}$$

and

$$|-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2} \quad \text{and}$$

$$\mathrm{Arg}(-1-i) = \arctan\left(\frac{-1}{-1}\right) = \frac{5\pi}{4} \quad \text{so that}$$

$$-1 - i = \sqrt{2}\, e^{i5\pi/4}$$