Calculus I Lecture 30

Example 4.25 Find $\int \frac{2x^3-3x^2-8x+24}{x^2-4}\,dx$

Solution:

$$\int \frac{2x^3-3x^2-8x+24}{x^2-4}\,dx = \int \frac{(x^2-4)(2x-3)+12}{(x^2-4)}\,dx$$

$$= \int \left(2x-3+\frac{12}{x^2-4}\right)dx = \int\left(2x-3+\frac{12}{(x+2)(x-2)}\right)dx$$

$$= \int\left(2x-3+\frac{-3}{x+2}+\frac{3}{x-2}\right)dx$$

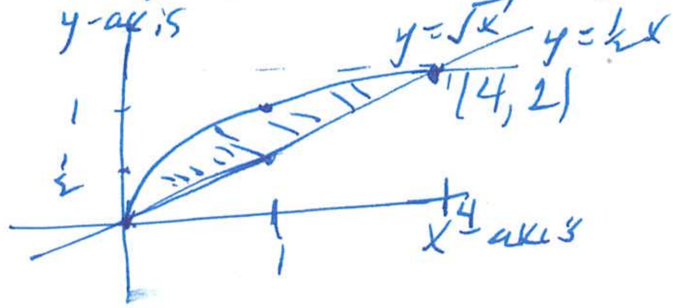
$$= x^2-3x-3\log(x+2)+3\log(x-2)+c$$

$$= x^2-3x+3\left(\log(x-2)-\log(x+2)\right)+c$$

$$= x^2-3x+3\log\left(\frac{x-2}{x+2}\right)+c$$

$$= x^2-3x+\log\left(\left(\frac{x-2}{x+2}\right)^3\right)+c,$$ where c is a constant.

Example 4.2 Find the area of the region enclosed by $y=\sqrt{x}$ and $y=\frac{1}{2}x$.



If $y=\sqrt{x}$ and $y=\frac{1}{2}x$ then $x^{\frac{1}{2}}=\frac{1}{2}x$ and $x=\frac{1}{4}x^2$ and $1=\frac{1}{4}x$ so that $x=4$ and $y=2$

$$\text{Area} = \int_{y=0}^{y=2} \left(\cancel{\tfrac{1}{2}x - \sqrt{x}}\right)(x_{right} - x_{left})\,dy$$

$$= \int_{y=0}^{y=2} (2y - y^2)\,dy = y^2 - \frac{1}{3}y^3 \Big]_{y=0}^{y=2}$$

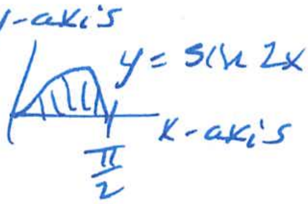
$$= \left(2^2 - \frac{1}{3}2^3\right) - \left(0^2 - \frac{1}{3}0^3\right) = 4 - \frac{8}{3} = \frac{4}{3}. \;//$$

<u>Example 4.27</u>

$$\int_0^{2\pi} \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + c \Big]_{x=0}^{x=2\pi}$$

$$= \left(-\frac{1}{2}\cos(4\pi) + c\right) - \left(-\frac{1}{2}\cos 0 + c\right)$$

$$= -\frac{1}{2} + c + \frac{1}{2} - c = 0.$$

Area = 4 times area 

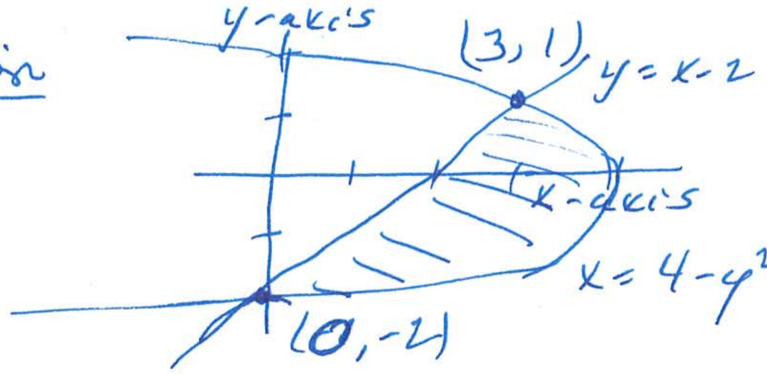
$$= 4\int_{x=0}^{x=\frac{\pi}{2}} (\sin 2x)\,dx = 4\left(-\frac{1}{2}\cos(2x) + c\right)\Big]_{x=0}^{x=\frac{\pi}{2}}$$

$$= 4\left(-\frac{1}{2}\cos\pi + c\right) - 4\left(-\frac{1}{2}\cos 0 + c\right)$$

$$= (-2)(-1) + 4c + 2\cdot 1 - 4c = 2 + 2 = 4. \;//$$

Example 4.28 Find the area enclosed by the curves $y = x - 2$ and $x = 4 - y^2$

Solution



If $y = x-2$ and $x = 4-y^2$ then

$$y+2 = 4-y^2 = (2-y)(2+y) \text{ so } y+2=0 \text{ or } 2-y=1.$$

So $y=-2$ or $y=1$. (If $y=1$ then $x = y+2 = 3$)

Then

$$\text{Area} = \int_{y=-2}^{y=1} (x_{right} - x_{left})\, dy$$

$$= \int_{y=-2}^{y=1} ((4-y^2) - (y+2))\, dy$$

$$= \int_{y=-2}^{y=1} (-y^2 - y + 2)\, dy$$

$$= -\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y \Big]_{y=-2}^{y=1}$$

$$= \left(-\frac{1}{3}\cdot 1 - \frac{1}{2}\cdot 1 + 2\cdot 1\right) - \left(-\frac{1}{3}(-8) - \frac{1}{2}(4) + 2(-2)\right)$$

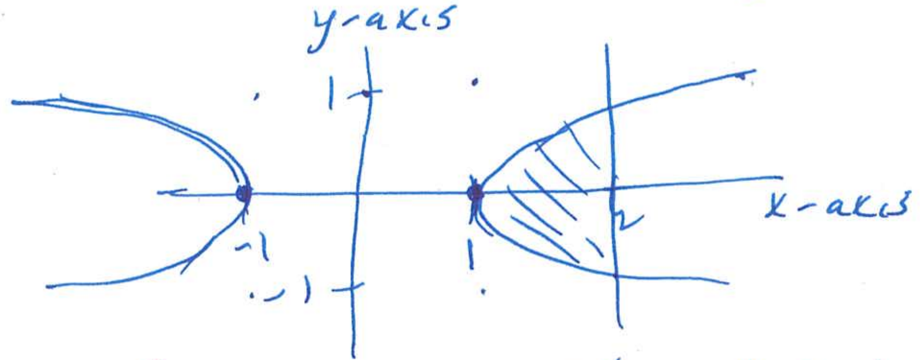
$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4 = -\frac{9}{3} - \frac{1}{2} + 8$$

$$= -3 - \frac{1}{2} + 8 = 5 - \frac{1}{2} = \frac{9}{2}\ //.$$

Example 4.29 Find the area enclosed by $x^2(1-y^2)=1$ and $x=2$.

$x^2(1-y^2)=1$ is $1-y^2=\frac{1}{x^2}$ is $y^2=1-\frac{1}{x^2}$

is $y=\pm\sqrt{1-\frac{1}{x^2}}$ or $x=\sqrt{\frac{1}{1-y^2}}$



If $x=2$ and $x^2(1-y^2)=1$ then $4(1-y^2)=1$

so that $1-y^2=\frac{1}{4}$ and $y^2=\frac{3}{4}$ and $y=\pm\sqrt{\frac{3}{4}}$

So Area $=\int_{y=-\frac{\sqrt{3}}{2}}^{y=\frac{\sqrt{3}}{2}}(x_{right}-x_{left})\,dy$

$=\int_{y=-\frac{\sqrt{3}}{2}}^{y=\frac{\sqrt{3}}{2}}\left(2-\sqrt{\frac{1}{1-y^2}}\right)dy$ Let $y=\sin\theta$

Then $\frac{dy}{d\theta}=\cos\theta$

$=\int_{y=-\frac{\sqrt{3}}{2}}^{y=\frac{\sqrt{3}}{2}}\left(2-\frac{1}{\sqrt{1-y^2}}\right)\frac{dy}{d\theta}\,d\theta$

$=\int_{y=-\frac{\sqrt{3}}{2}}^{y=\frac{\sqrt{3}}{2}}\left(2-\frac{1}{\sqrt{1-\sin^2\theta}}\right)\cos\theta\,d\theta$

$=\int_{y=-\frac{\sqrt{3}}{2}}^{y=\frac{\sqrt{3}}{2}}\left(2-\frac{1}{\cos\theta}\right)\cos\theta\,d\theta$

$=2y\Big]_{y=-\sqrt{3}/2}^{y=\sqrt{3}/2}-\theta\Big]_{\theta=\arcsin(-\sqrt{3}/2)}^{\theta=\arcsin(\sqrt{3}/2)}$

$$= \left(2\frac{\sqrt{3}}{2} - 2\left(-\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{\pi}{3} - \left(\frac{-\pi}{3}\right)\right)$$

$$= \sqrt{3} + \sqrt{3} - \frac{\pi}{3} - \frac{\pi}{3} = 2\sqrt{3} - \frac{2\pi}{3}$$