Calculus I Lecture 29

Example 4.14 Find $\int \frac{1}{2x-3}\,dx$

Solution: $\int \frac{1}{2x-3}\,dx = \frac{1}{2}\log(2x-3) + c$, where c is a constant

because

$$\frac{d}{dx}\left(\frac{1}{2}\log(2x-3)+c\right) = \frac{1}{2}\cdot\frac{1}{2x-3}\cdot\frac{d}{dx}(2x-3)+0$$

$$= \frac{1}{2}\cdot\frac{1}{2x-3}\cdot 2 = \frac{1}{2x-3}.$$

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Example 4.16 Find $\int \frac{1}{x^2+2x+2}\,dx$

Solution:

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{x^2+2x+1+1}\,dx$$

$$= \int \frac{1}{(x+1)^2+1}\,dx.$$ Let $u = x+1$ so that $\frac{du}{dx} = 1$.

Then

$$\int \frac{1}{(x+1)^2+1}\,dx = \int \frac{1}{u^2+1}\frac{du}{dx}\,dx = \int \frac{1}{u^2+1}\,du$$

$$= \arctan(u) + c = \arctan(x+1) + c,$$ where c is a constant.

Example 4.17 Find $\int \frac{x+1}{x^2+2x+2}\,dx$.

Solution Let $u = x+1$. Then $\frac{du}{dx} = 1$.

$$\int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{x+1}{x^2+2x+1+1}\,dx = \int \frac{x+1}{(x+1)^2+1}\,dx$$

$$= \int \frac{u}{u^2+1}\frac{du}{dx}\,dx = \int \frac{u}{u^2+1}\,du. \quad \text{Let } v = u^2+1$$

$$\frac{dv}{du} = 2u.$$

So

$$\int \frac{u}{u^2+1}\,du = \int \frac{1}{v}\,\frac{1}{2}\frac{dv}{du}\,du = \int \frac{1}{2}\cdot\frac{1}{v}\,dv$$

$$= \frac{1}{2}\log(v) + c = \frac{1}{2}\log(u^2+1) + c$$

$$= \frac{1}{2}\log((x+1)^2+1) + c = \frac{1}{2}\log(x^2+2x+2) + c,$$

where c is a constant.

Example 4.18 Find $\int \frac{x-2}{x^2+2x+2}\,dx$

Solution

$$\int \frac{x-2}{x^2+2x+2}\,dx = \int\left(\frac{x+1}{x^2+2x+2} - 3\,\frac{1}{x^2+2x+2}\right)dx$$

$$= \frac{1}{2}\log(x^2+2x+2) - 3\arctan(x+1) + c,$$ where c is a constant.

This solution comes from combining the solutions of Example 4.16 and Example 4.17.

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Example 4.19 Find $\int \frac{2x+1}{x^2+2x+1}\, dx$.

Solution:

$$\int \frac{2x+1}{x^2+2x+1}\, dx = \int \frac{2(x+1)-1}{(x+1)^2}\, dx$$

$$= \int \left(2\frac{1}{x+1} - \frac{1}{(x+1)^2}\right) dx = \int \left(2\cdot\frac{1}{x+1} - (x+1)^{-2}\right) dx$$

$$= 2\log(x+1) - \frac{1}{-3}(x+1)^{-3} + c$$

$$= 2\log(x+1) + \frac{1}{3}(x+1)^{-3} + c,$$ where c is a constant. //

Example 4.21 Find $\int \frac{9x+1}{(x-3)(x+1)}\, dx$

Solution: $\frac{9x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$ with

$A(x+1) + B(x-3) = 9x+1$ so that

$A+B=9$ and $A-3B=1$.

So $B = 9-A$ and $A - 3(9-A) = 1$.

So $4A - 27 = 1$. So $4A = 28$. So $A = 7$.

So $B = 9 - A = 9 - 7 = 2$. So

$$\frac{9x+1}{(x-3)(x+1)} = \frac{7}{x-3} + \frac{2}{x+1}.$$

So $\displaystyle\int \frac{9x+1}{(x-3)(x+1)}\,dx = \int\left(\frac{7}{x-3} + \frac{2}{x+1}\right)dx$

$= 7\log(x-3) + 2\log(x+1) + c$, where c is a constant. //

<u>Example 4.22</u> Find $\displaystyle\int \frac{3x^2-2x+1}{(x+1)(x^2+2x+2)}\,dx$

<u>Solution</u>: $\displaystyle\frac{3x^2-2x+1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$ with

$Ax^2+2Ax+2A+Bx^2+Bx+Cx+C = 3x^2-2x+1$,

so that

$$\begin{aligned} A+B&=3 \\ 2A+B+C&=-2 \\ 2A+C&=1. \end{aligned} \quad \text{So} \quad \begin{aligned} A&=3-B \\ B+1&=-2 \\ C&=1-2A. \end{aligned} \quad \text{So} \quad \begin{aligned} A&=3+3=6 \\ B&=-3 \\ C&=1-2\cdot 6=-1 \end{aligned}$$

So

$$\int \frac{3x^2-2x+1}{(x+1)(x^2+2x+2)}\,dx = \int\left(\frac{6}{x+1} + \frac{-3x-11}{x^2+2x+2}\right)dx$$

$$= \int\left(\frac{6}{x+1} + \frac{-3(x+1)-8}{x^2+2x+2}\right)dx$$

$$= 6\log(x+1) - 3\cdot\tfrac{1}{2}\log(x^2+2x+2) - 8\arctan(x+1) + c,$$

where c is a constant. This solution is obtained by using the solutions of Example 4.16 and Example 4.17.

<u>Example 4.23</u> Find $\int \frac{3x+1}{x^2+4x+4}\,dx$

<u>Solution</u>: $\int \frac{3x+1}{x^2+4x+4}\,dx = \int \frac{3x+1}{(x+2)^2}\,dx$

$$= \int \frac{3(x+2)-5}{(x+2)^2}\,dx = \int 3\frac{1}{(x+2)} - 5\frac{1}{(x+2)^2}\,dx$$

$$= \int \left(3\cdot\frac{1}{x+2} - 5(x+2)^{-2}\right)dx$$

$$= 3\log(x+2) - \frac{5}{-1}(x+2)^{-1} + c$$

$$= 3\log(x+2) + \frac{5}{x+2} + c,$$ where c is a constant.

<u>Example 4.24</u> Find $\int \frac{3x+1}{x^2+4x+4}\,dx$

<u>Solution</u> $\int \frac{3x+1}{x^2+4x+4}\,dx = \int \frac{3x+1}{(x+2)^2}\,dx.$

Let $u = x+2$. Then $\frac{du}{dx} = 1$.

$$\int \frac{3x+1}{(x+2)^2}\,dx = \int \frac{3(u-2)+1}{u^2}\frac{du}{dx}\,dx = \int \frac{3u-5}{u^2}\,du$$

$$= \int\left(3\frac{1}{u} - 5u^{-2}\right)du = 3\log(u) - 5\frac{u^{-1}}{-1} + c$$

$$= 3\log(x+2) + 5(x+2)^{-1} + c,$$ where c is a constant.