Example 4.1 Evaluate $\int_0^1 x^2\,dx$ using

$$1^2+2^2+\cdots+(n-1)^2=\frac{n(n-1)(2n-1)}{6}$$

Solution:

$$\int_0^1 x^2\,dx = \lim_{N\to\infty}\left(\frac{1}{N}0^2+\frac{1}{N}\left(\frac{1}{N}\right)^2+\frac{1}{N}\left(\frac{2}{N}\right)^2+\cdots+\frac{1}{N}\left(\frac{N-1}{N}\right)^2\right)$$

$$=\lim_{N\to\infty}\frac{1}{N^3}\left(1^2+2^2+\cdots+(N-1)^2\right)$$

$$=\lim_{N\to\infty}\frac{1}{N^3}\cdot\frac{N(N-1)(2N-1)}{6}=\lim_{N\to\infty}\frac{1\left(1-\frac{1}{N}\right)\left(2-\frac{1}{N}\right)}{6}$$

$$=\frac{1\cdot1\cdot2}{6}=\frac{1}{3}.$$

Note: $\int x^2\,dx\Big]_{x=0}^{x=1}=\left(\frac{1}{3}x^3+c\right)\Big]_{x=0}^{x=1}=\left(\frac{1}{3}\cdot1^3+c\right)-\left(\frac{1}{3}\cdot0^3+c\right)$

$=\frac{1}{3}$, which is Example 4.3.

Example 4.5 Find $\int\left(5e^{2x}-\frac{3}{x}\right)dx$.

Solution: $\int\left(5e^{2x}-\frac{3}{x}\right)dx=5\int e^{2x}dx-3\int\frac{1}{x}dx$

$=5\,\frac{1}{2}e^{2x}-3\log x+c$, where c is a constant,

since

$$\frac{d}{dx}\left(\frac{5}{2}e^{2x}-3\log x+c\right)=\frac{5}{2}e^{2x}\cdot2-3\,\frac{1}{x}+0=5e^{2x}-\frac{3}{x}.$$

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Example 4.6 Find $\int_1^{e^{\pi/2}} \frac{\sin(\log x)}{x}\,dx$

Solution Let $u = \log x$. Then $\frac{du}{dx} = \frac{1}{x}$.

$$\int_{x=1}^{x=e^{\pi/2}} \frac{\sin(\log x)}{x}\,dx = \int_{u=\log 1}^{u=\log e^{\pi/2}} \sin u \,\frac{du}{dx}\,dx = \int_{u=0}^{u=\frac{\pi}{2}} \sin u\,du$$

$$= \int \sin u\,du \Big]_{u=0}^{u=\frac{\pi}{2}} = -\cos u + c\Big]_{u=0}^{u=\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0)$$

$$= -0 + 1 = 1.$$

Example 4.7 Rewrite $\int_0^1 2x e^{-x^2}\,dx$

Solution Let $u = -x^2$. Then $\frac{du}{dx} = -2x$.

$$\int_{x=0}^{x=1} 2x e^{-x^2}\,dx = \int_{u=-0^2}^{u=-1^2} e^u\left(-\frac{du}{dx}\right)dx = \int -e^u\,du\Big]_{u=0}^{u=-1}$$

$$= -e^u + c\Big]_{u=0}^{u=-1} = (-e^{-1} + c) - (-e^0 + c) = -\frac{1}{e} + 1.$$

Example 4.8 Find $\int 2x(x^2-5)^4\,dx$

Solution: Let $u = x^2 - 5$. Then $\frac{du}{dx} = 2x$ and

$$\int 2x(x^2-5)^4\,dx = \int u^4 \frac{du}{dx}\,dx = \int u^4\,du = \frac{u^5}{5} + c$$

$$= \frac{1}{5}(x^2-5)^5 + c,$$ where c is a constant.

Example 4.9 $\int \cos(3x)\sqrt{\sin(3x)+4}\,dx$.

Solution: Let $u = \sin(3x)+4$. Then $\frac{du}{dx} = \cos(3x)\cdot 3$.

$$\int \cos(3x)(\sin(3x)+4)^{\frac{1}{2}}dx = \int u^{\frac{1}{2}}\frac{du}{dx}dx = \int u^{\frac{1}{2}}du$$

$$= \frac{2}{3}u^{3/2} + c = \frac{2}{3}(\sin(3x)+4)^{3/2} + c,$$

where c is a constant. //

Example 4.10 $\int (2x+1)\sqrt{x-3}\,dx$

Solution $\int (2x+1)(x-3)^{\frac{1}{2}}dx = \int (2(x-3)+7)(x-3)^{\frac{1}{2}}dx$

$$= \int \left(2(x-3)^{3/2} + 7(x-3)^{\frac{1}{2}}\right)dx$$

$$= 2\cdot\frac{2}{5}(x-3)^{5/2} + 7\cdot\frac{2}{3}(x-3)^{3/2} + c$$

$$= \frac{4}{5}(x-3)^{5/2} + \frac{14}{3}(x-3)^{3/2} + c,$$ where c is a constant.

Example 4.11 Find $\int \frac{2x}{(x+1)^{10}}\,dx$

Solution $\int \frac{2x}{(x+1)^{10}}\,dx = \int (2(x+1)-2)(x+1)^{-10}\,dx$

$= \int (2(x+1)^{-9} - 2(x+1)^{-10})\,dx$

$= \frac{2}{-8}(x+1)^{-8} - \frac{2}{-9}(x+1)^{-9} + c = -\frac{1}{4}(x+1)^{-8} + \frac{2}{9}(x+1)^{-9} + c,$

where c is a constant.

Example 4.12 Find $\int \sin^{25}x \cos^3 x\,dx$

Solution $\int \sin^{25}x\cos^3x\,dx = \int \sin^{25}x \cos^2 x \cos x\,dx$

$= \int \sin^{25}x(1-\sin^2 x)\cos x\,dx$

$= \int (\sin^{25}x - \sin^{27}x)\cos x\,dx$

$= \frac{1}{26}\sin^{26}x - \frac{1}{28}\sin^{28}x + c$, where c is a constant. //

Example 4.13 Find $\int \sin^4 x \, dx$.

Solution $\int \sin^4 x\,dx = \int \left(\frac{1}{2i}(e^{ix}-e^{-ix})\right)^4 dx = \frac{1}{16}\int (e^{ix}-e^{-ix})^4 dx$

$= \frac{1}{16}\int (e^{4ix} - 4e^{i3x}e^{-ix} + 6e^{i2x}e^{-i2x} - 4e^{ix}e^{-3ix} + e^{-4ix})\,dx$

$= \frac{1}{16}\int (e^{i4x} + e^{-i4x} - 4(e^{i2x}+e^{-i2x}) + 6)\,dx$

$= \frac{1}{16}\left(\frac{1}{4i}e^{i4x} + \frac{1}{-4i}e^{-i4x} - 4\left(\frac{1}{2i}e^{i2x} + \frac{1}{-2i}e^{-i2x}\right) + 6x\right) + c$

$= \frac{1}{16\cdot 2}\left(\frac{1}{2i}(e^{i4x}-e^{-i4x})\right) - \frac{1}{4}\left(\frac{1}{2i}(e^{i2x}-e^{-i2x})\right) + \frac{3}{8}x + c$

$= \frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$, where c is a constant. //