Example Find the vertical asymptotes of

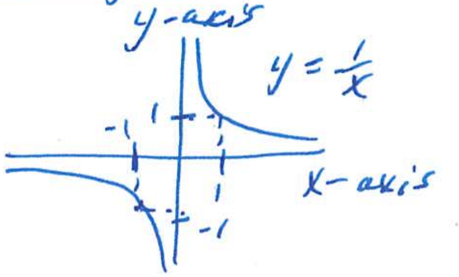
$$f(x) = \frac{x^2+2x+1}{x^2-1}$$

Solution

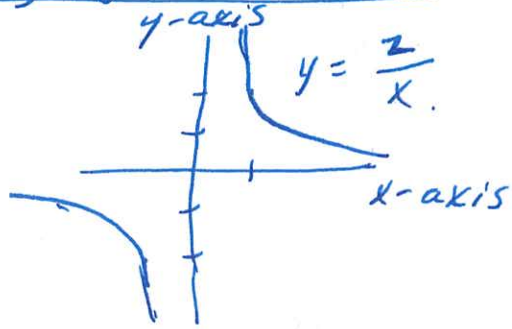
$$\frac{x^2+2x+1}{x^2-1} = \frac{(x+1)^2}{(x-1)(x+1)} = \frac{x+1}{x-1} = \frac{x-1+2}{x-1}$$

$$= 1 + \frac{2}{x-1}$$

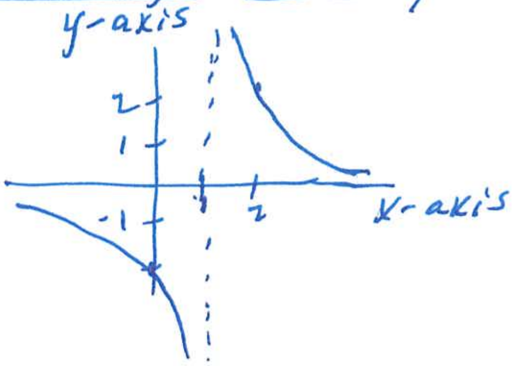
Basic graph



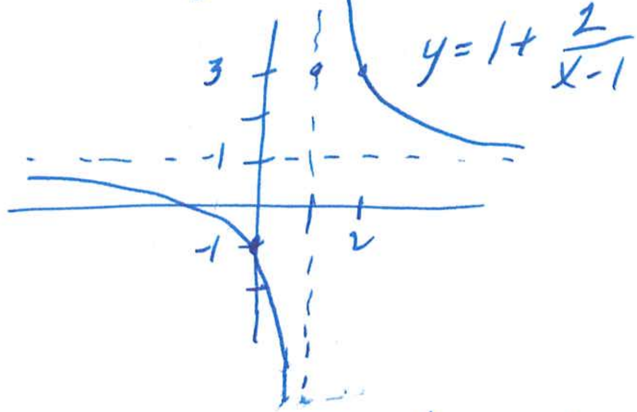
Streach y by factor of 2



shift x by 1 to the right

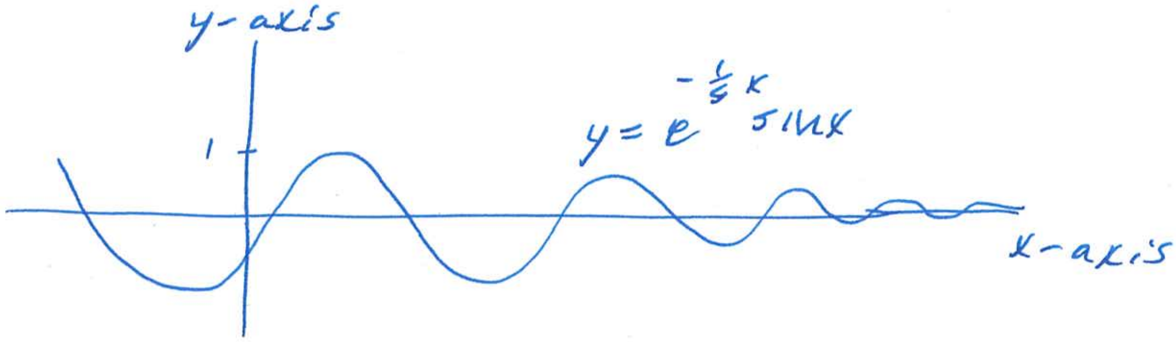


shift y by 1 up



This graph has a vertical asymptote at $x=1$ (as $y \to \pm\infty$)
and a horizontal asymptote at $y=1$ (as $x \to \pm\infty$).

Example 3.27 The horizontal asymptote of $f(x) = e^{-\frac{1}{5}x}\sin(x)$ is $y = 0$ (as $x \to \infty$).



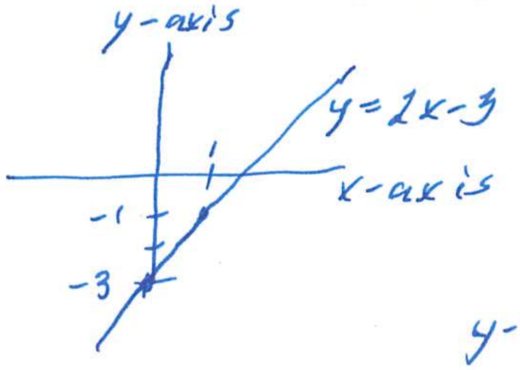
Example 3.28 Find the asymptotes of

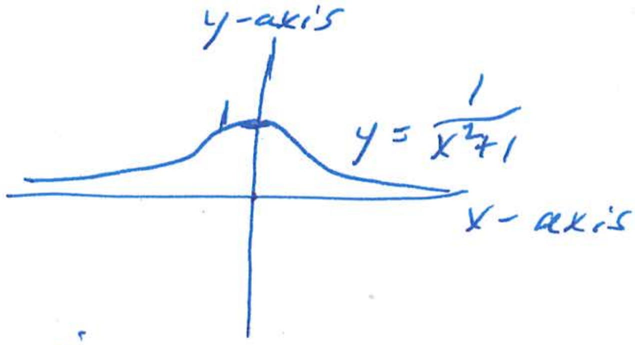
$$f(x) = \frac{2x^3 - 3x^2 + 2x - 2}{x^2+1}$$

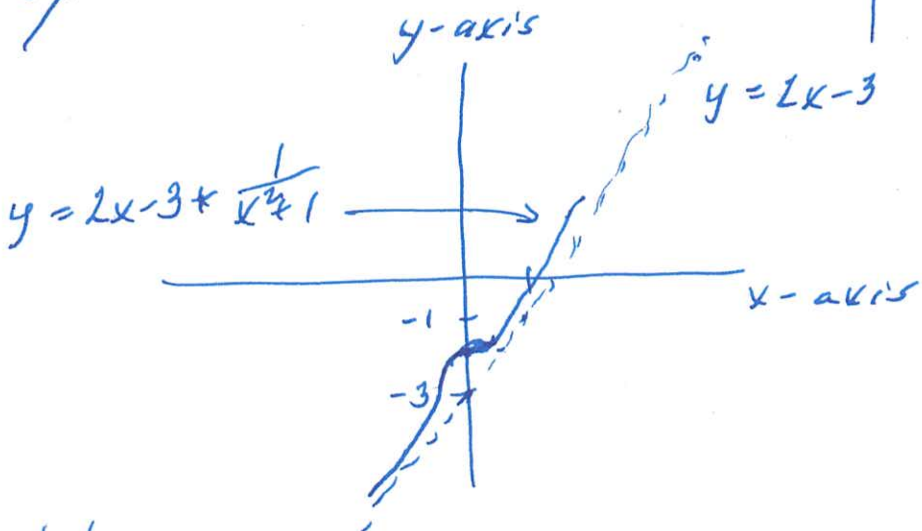
Solution

$$\frac{2x^3-3x^2+2x-2}{x^2+1} = \frac{(x^2+1)(2x-3)+1}{x^2+1}$$

$$= 2x - 3 + \frac{1}{x^2+1}$$





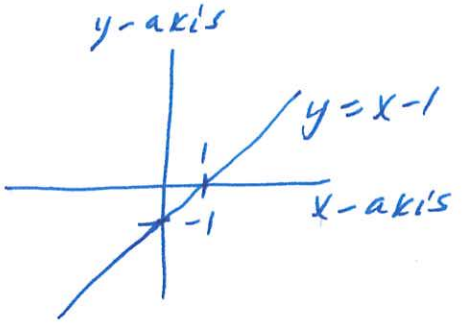


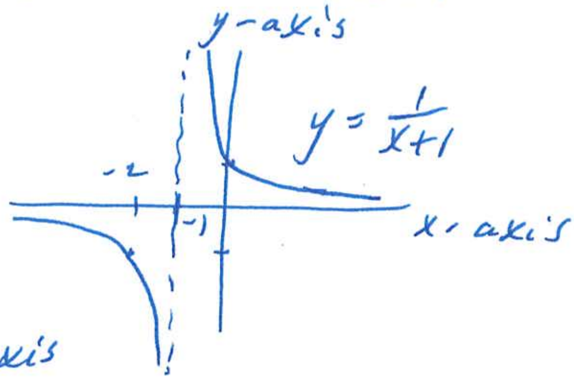
Asymptote $y = 2x - 3$ as $x \to \pm\infty$.

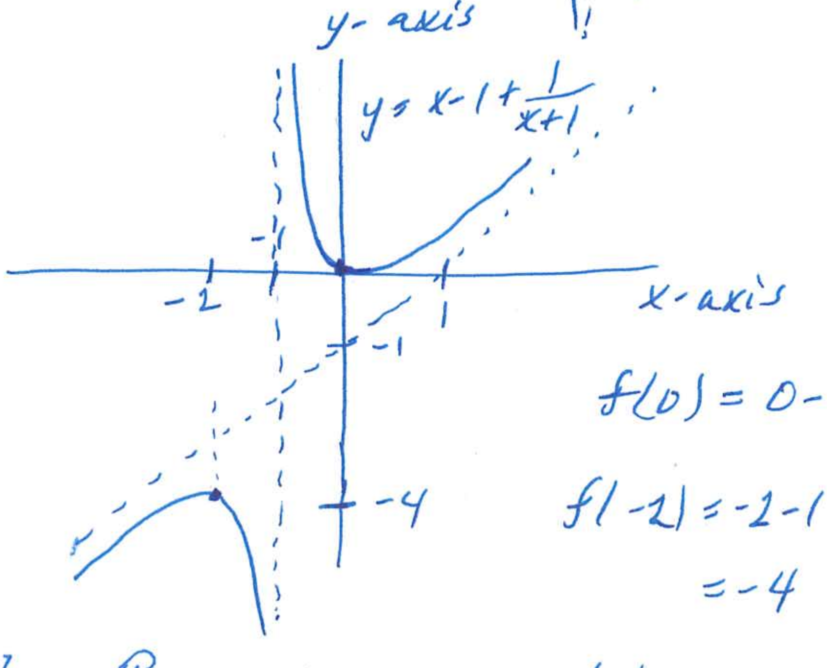
3.29
Example Graph and analyze $f(x) = \dfrac{x^2}{x+1}$

Solution $\dfrac{x^2}{x+1} = \dfrac{x(x+1)-x}{x+1} = x - \dfrac{x}{x+1}$

$= x - \dfrac{x+1-1}{x+1} = x - 1 + \dfrac{1}{x+1}$







$f(0) = 0 - 1 + \dfrac{1}{0+1} = 0$

$f(-2) = -2 - 1 + \dfrac{1}{-2+1} = -3 - 1$
$= -4$

$f : \mathbb{R} - \{-1\} \to \mathbb{R}$ has asymptotes

$x \mapsto x - 1 + \dfrac{1}{x-1}$

(a) $y = -1$ as $x \to -1$

(b) $y = x - 1$ as $x \to \pm\infty$

If $x = 0$ then $f(0) = 0 - 1 + \dfrac{1}{0+1} = -1 + 1 = 0$.

If $y = 0$ then $0 = \dfrac{x^2}{x+1}$ so $x^2 = 0$ and $x = 0$

So $(x,y) = (0,0)$ is the only x-intercept and the only y-intercept.

$$\frac{df}{dx} = \frac{d\left(x-1+\frac{1}{x+1}\right)}{dx} = 1-0+(-1)(x+1)^{-2}$$

$$= 1 - \frac{1}{(x+1)^2}$$

If $\frac{df}{dx} > 0$ then $1 - \frac{1}{(x+1)^2} > 0$ and $1 > \frac{1}{(x+1)^2}$

and $(x+1)^2 > 1$ and $x+1 > 1$ or $x+1 < -1$

So, if $\frac{df}{dx} > 0$ then $x > 0$ or $x < -2$.

If $\frac{df}{dx} < 0$ then $-1 < x < 0$ or $-2 < x < -1$.

So f is increasing for $x \in \mathbb{R}_{>0} \cup \mathbb{R}_{<-2}$ and
f is decreasing for $x \in \mathbb{R}_{(-2,0)} - \{-1\}$.

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(1 - \frac{1}{(x+1)^2}\right) = 0 - (-2)(x+1)^{-3} = \frac{2}{(x+1)^3}.$$

If $\frac{d^2f}{dx^2} > 0$ then $\frac{2}{(x+1)^3} > 0$ and $(x+1)^3 > 0$

so that $x > -1$.

If $\frac{d^2f}{dx^2} < 0$ then $\frac{2}{(x+1)^3} < 0$ and $(x+1)^3 < 0$

so that $x < -1$.

So f is concave up for $x \in \mathbb{R}_{(-1,\infty)}$
and concave down for $x \in \mathbb{R}_{(-\infty,-1)}$.

If $\frac{df}{dx} = 0$ then $1 - \frac{1}{(x+1)^2} = 0$ and $1 = \frac{1}{(x+1)^2}$

and $(x+1)^2 = 1$ and $(x+1) = \pm 1$ so that

$x = 0$ or $x = 2$.

f has a local minimum at $x = 0$
and a local maximum at $x = 2$.

There are no global maxima or minima of f.

Since $f(x)$ is not ~~define~~ a real number
at $x = -1$ then f has no point of inflection. //