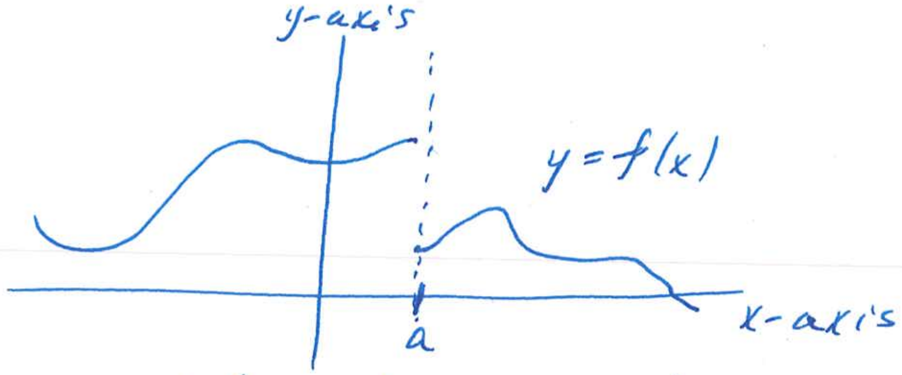
Calculus 1 Lecture 20

A function $f(x)$ is <u>continuous</u> at $x=a$ if the graph doesn't jump at $x=a$,

i.e. if $\lim_{x\to a} f(x) = f(a)$

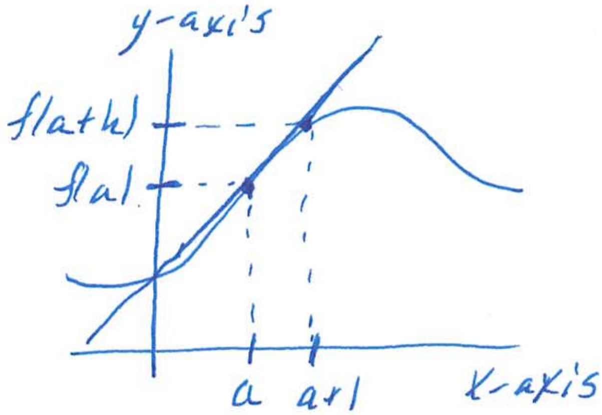


Not continuous at $x=a$

Think about

$$D_a f = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

in terms of the graph

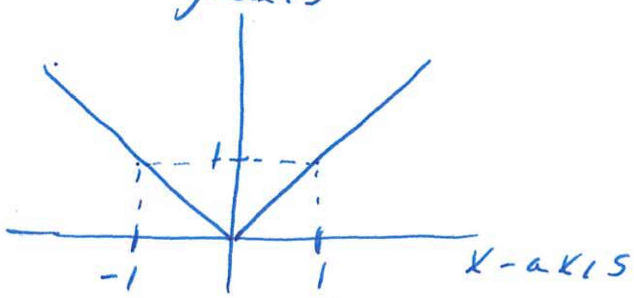


$$\frac{f(a+h)-f(a)}{h} = \frac{\text{change in } f}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \text{slope of line connecting } (a, f(a)) \text{ and } (a+h, f(a+h))$$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \text{slope of } f \text{ at the point } x=a.$$

A function $f(x)$ is <u>differentiable</u> at $x=a$ if the derivative $D_a f$ is a real number, i.e. if the slope of the graph of $f(x)$ at $x=a$ is a real number.

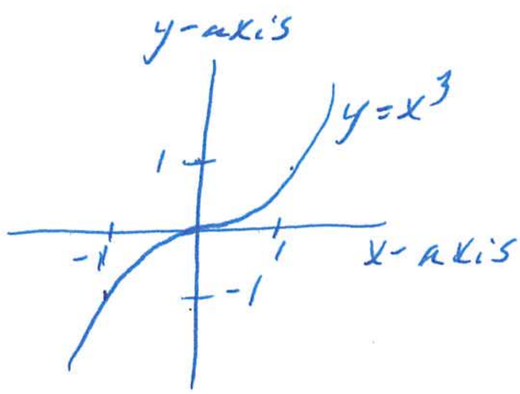
<u>Example</u> Graph $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

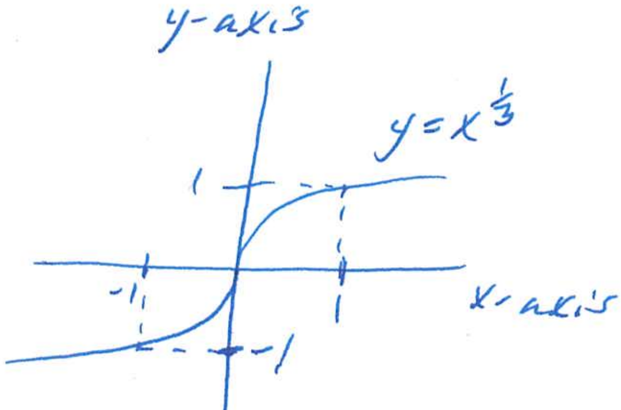


Then $D_a f = \begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \\ ?, & \text{if } a = 0 \end{cases}$

So f is <u>not</u> differentiable at $a=0$.

<u>Example</u> Graph $y = x^{\frac{1}{3}}$





<u>Note</u>: $y = x^{1/3}$ is the same as $y^3 = x$.

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$ and $D_0 y = \left.\frac{dy}{dx}\right|_{x=0}$ is not a real number.

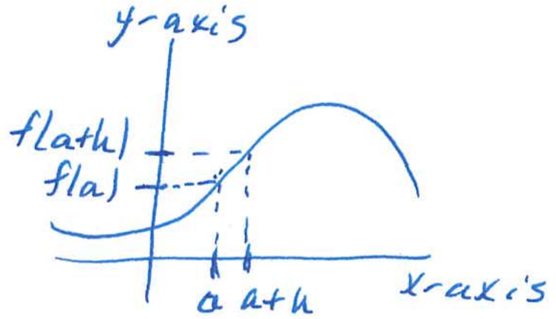
So $f(x) = x^{1/3}$ is <u>not</u> differentiable at $x = 0$.

A function is <u>increasing</u> at $x=a$ if it is going up at $x=a$,

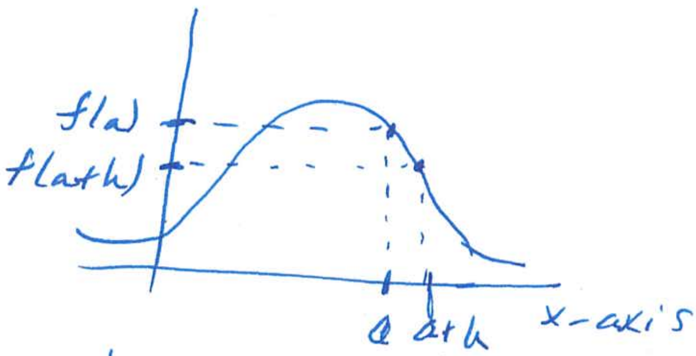
i.e. if $f(a+h) > f(a)$ for small $h > 0$,

i.e. if slope is positive,

i.e. if $D_a f > 0$.



increasing at $x=a$.



decreasing at $x=a$

A function $f(x)$ is <u>decreasing</u> at $x=a$ if it is going down at $x=a$,

i.e. if $f(a+h) < f(a)$ for small $h > 0$

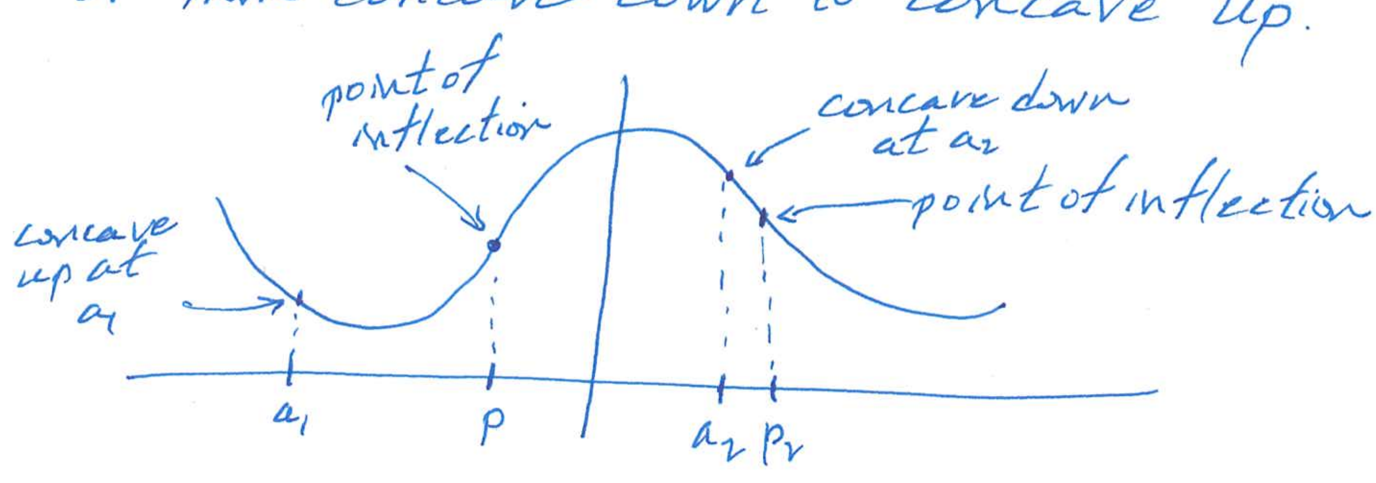
i.e. if the slope is negative at $x=a$,

i.e. if $D_a f < 0$.

f is <u>concave up</u> at $x=a$ if it is right side up bowl shaped at $x=a$.
i.e. if the slope of f is getting larger at $x=a$,
i.e. if $\frac{df}{dx}$ is increasing at $x=a$
i.e. if $\left.\frac{d^2f}{dx^2}\right|_{x=a} > 0$

f is <u>concave down</u> at $x=a$ if it is upside down bowl shaped at $x=a$
i.e. if the slope of f is getting smaller at $x=a$,
i.e. if $\frac{df}{dx}$ is decreasing at $x=a$
i.e. if $\left.\frac{d^2f}{dx^2}\right|_{x=a} < 0$

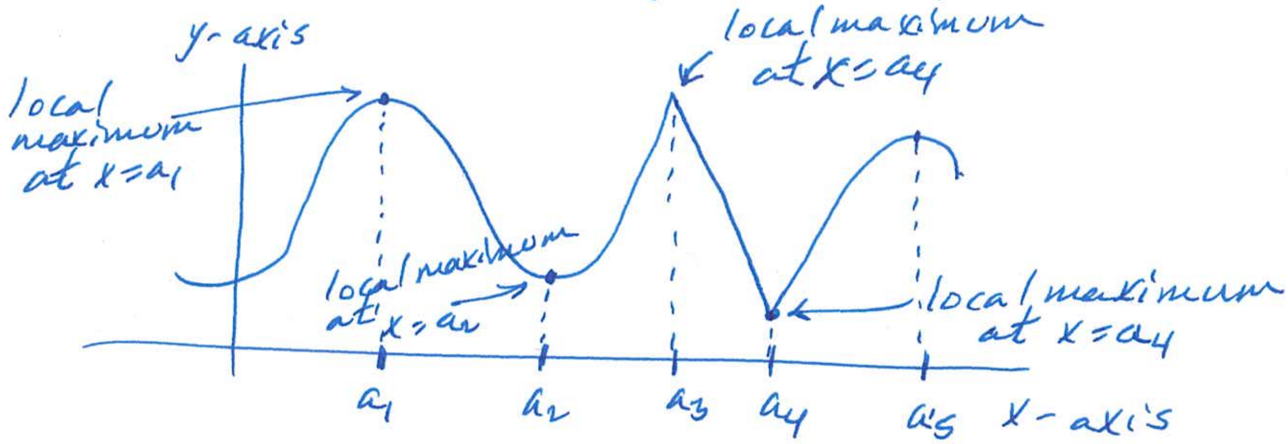
A <u>point of inflection</u> is a point where f changes from concave up to concave down, or from concave down to concave up.



A <u>local maximum</u> is a point $x=a$ where $f(a)$ is bigger than the $f(x)$ around it.

A <u>local minimum</u> is a point $x=a$ where $f(a)$ is smaller than the $f(x)$ around it

i.e. $f(a) < f(a+h)$ for small h.



A <u>critical point</u> is a point where a maximum or minimum might occur.

<u>Note</u>: (1) If $f(x)$ is continuous and differentiable and $x=a$ is a maximum then

$$\left.\frac{df}{dx}\right|_{x=a}=0 \quad \text{and} \quad \left.\frac{d^2f}{dx^2}\right|_{x=a}<0$$

(2) If $f(x)$ is continuous at $x=a$,
$f(x)$ is differentiable at $x=a$,

$$\left.\frac{df}{dx}\right|_{x=a}=0 \quad \text{and} \quad \left.\frac{d^2f}{dx^2}\right|_{x=a}>0 \text{ then}$$

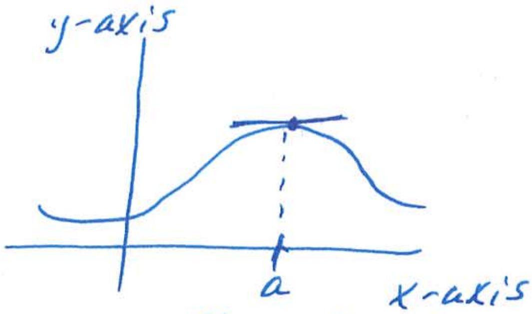
$x=a$ is a minimum.

Where can a maximum or minimum occur?

(a) A point where $f(x)$ is differentiable and

$$\left.\frac{df}{dx}\right|_{x=a} = 0.$$

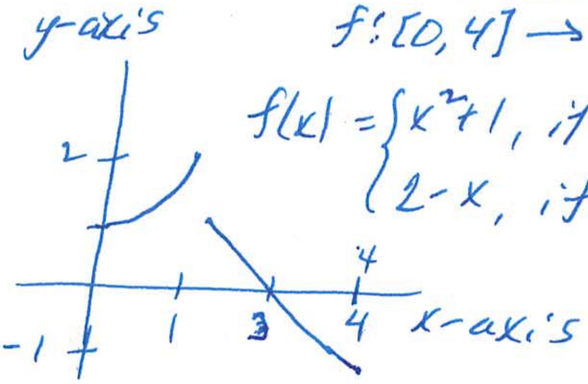
(b) A point $x=a$ where $f(x)$ is not continuous.



$f(x)$ differentiable and $\left.\frac{df}{dx}\right|_{x=a} = 0$

$f: [0,4] \to \mathbb{R}$

$$f(x) = \begin{cases} x^2+1, & \text{if } 0 \le x \le 1 \\ 2-x, & \text{if } 1 < x \le 4 \end{cases}$$



$x=1$ is a maximum

$x=4$ is a minimum.

(c) A point on the boundary of where $f(x)$ is defined.