Calculus I Lecture 19 10.09.2019

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Properties of $\frac{d}{dx}$

(1) $\frac{dx}{dx} = 1$

(2) If $c \in \mathbb{C}$ then $\frac{d(cf)}{dx} = c\frac{df}{dx}$

(3) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$

(4) $\frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx}g$

(5) $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$

Theorem 0 $\frac{d1}{dx} = 0$.

Proof $\frac{d1}{dx} = \frac{d(1\cdot 1)}{dx} = \frac{d1}{dx}\cdot 1 + 1\cdot\frac{d1}{dx}$.

Subtract $\frac{d1}{dx}$ from both sides to get $\frac{d1}{dx} = 0$.

Theorem 2: $\frac{dx^2}{dx} = 2x$.

Proof $\frac{dx^2}{dx} = \frac{d(x\cdot x)}{dx} = x\frac{dx}{dx} + \frac{dx}{dx}x = x + x = 2x$. //

Theorem 3 $\frac{dx^3}{dx} = 3x^2$.

Proof $\frac{dx^3}{dx} = \frac{d(x^2 \cdot x)}{dx} = x^2 \frac{dx}{dx} + \frac{dx^2}{dx} \cdot x = x^2 + 2x \cdot x$

$= 3x^2$. //

Theorem 4 $\frac{dx^4}{dx} = 4x^3$.

Proof $\frac{dx^4}{dx} = \frac{d(x^3 \cdot x)}{dx} = x^3 \frac{dx}{dx} + \frac{dx^3}{dx} \cdot x = x^3 + 3x^2 \cdot x$

$= 4x^3$. //

Similarly for the proofs of Theorems 5, 6, 7, 8, ...

Theorem e Let

$$e^x = 1 + x + \frac{1}{1 \cdot 2} x^2 + \frac{1}{1 \cdot 2 \cdot 3} x^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \ldots$$

Then $\frac{de^x}{dx} = e^x$.

Proof $\frac{de^x}{dx} = \frac{d}{dx}\left(1 + x + \frac{1}{1 \cdot 2} x^2 + \frac{1}{1 \cdot 2 \cdot 3} x^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \ldots\right)$

$= \frac{d1}{dx} + \frac{dx}{dx} + \frac{1}{1 \cdot 2} \frac{dx^2}{dx} + \frac{1}{1 \cdot 2 \cdot 3} \frac{dx^3}{dx} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \frac{dx^4}{dx} + \ldots$

$= 0 + 1 + \frac{1}{1 \cdot 2} 2x + \frac{1}{1 \cdot 2 \cdot 3} 3x^2 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} 4x^3 + \ldots$

$= 1 + x + \frac{1}{1 \cdot 2} x^2 + \frac{1}{1 \cdot 2 \cdot 3} x^3 + \ldots = e^x$. //

Theorem log Let log be such that

$$\log(e^x) = x \text{ and } e^{\log x} = x.$$

Then $\dfrac{d \log x}{dx} = \dfrac{1}{x}$.

Proof Let $y = \log x$. Then $e^y = x$. So

$$\frac{de^y}{dx} = \frac{dx}{dx}. \quad \text{So} \quad e^y \frac{dy}{dx} = 1.$$

So $\dfrac{dy}{dx} = \dfrac{1}{e^y}$. So $\dfrac{d(\log x)}{dx} = \dfrac{1}{x}$. //

Theorem x^a Let $a \in \mathbb{C}$ and let $x^a = e^{a \log x}$.

~~Theorem~~ Then $\dfrac{dx^a}{dx} = ax^{a-1}$.

Proof
$$\frac{dx^a}{dx} = \frac{d\, e^{a\log x}}{dx} = e^{a\log x} \frac{d\, a\log x}{dx}$$
$$= x^a a \frac{1}{x} = ax^{a-1}. //$$

Theorem sin and cos Let

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \text{ and } \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

Then

$$\frac{d(\sin x)}{dx} = \cos x \text{ and } \frac{d(\cos x)}{dx} = -\sin x.$$

Proof $\frac{d\sin x}{dx} = \frac{d}{dx}\frac{1}{2i}(e^{ix} - e^{-ix}) = \frac{1}{2i}\left(\frac{de^{ix}}{dx} - \frac{de^{-ix}}{dx}\right)$

$= \frac{1}{2i}\left(ie^{ix} - (-i)e^{-ix}\right) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x,$

and

$\frac{d\cos x}{dx} = \frac{d}{dx}\frac{1}{2}(e^{ix} + e^{-ix}) = \frac{1}{2}\left(\frac{de^{ix}}{dx} + \frac{de^{-ix}}{dx}\right)$

$= \frac{1}{2}\left(ie^{ix} + (-i)e^{-ix}\right) = \frac{1}{2}i\left(e^{ix} - e^{-ix}\right)$

$= \frac{-1}{i^2}\cdot\frac{1}{2}\cdot i\,(e^{ix} - e^{-ix}) = \frac{-1}{2i}(e^{ix} - e^{-ix})$

$= -\sin x.$ //

Example 3.3 Let $f(x)=2e^x+3x^{-7}$. Find $\frac{df}{dx}$.

Solution $\frac{df}{dx}=\frac{d(2e^x+3x^{-7})}{dx}=2\frac{de^x}{dx}+3\frac{dx^{-7}}{dx}$

$=2e^x+3(-7)x^{-7-1}=2e^x-21x^{-8}$. //

Example 3.5 Let $f=(x^4-3x)\log x$. Find $\frac{df}{dx}$.

Solution $\frac{df}{dx}=\frac{d((x^4-3x)\log x)}{dx}=$

$=(x^4-3x)\frac{1}{x}+(4x^3-3)\log x=x^3-3+4x^3\log x-3\log x$ //

Example 3.7 Let $y=\frac{x^3}{x^2+1}$. Find $\frac{dy}{dx}$.

Solution: $\frac{dy}{dx}=\frac{d}{dx}(x^3(x^2+1)^{-1})=x^3(-1)(x^2+1)^{-2}2x$

$+3x^2(x^2+1)^{-1}$

$=\frac{-2x^4}{(x^2+1)^2}+\frac{3x^2}{(x^2+1)}=\frac{-2x^4+3x^2(x^2+1)}{(x^2+1)^2}$

$=\frac{-2x^4+3x^4+1}{(x^2+1)^2}=\frac{x^4+1}{(x^2+1)^2}$. //

Example 3.8 Let $y = \tan x$. Find $\frac{dy}{dx}$

Solution: $\frac{dy}{dx} = \frac{d\left(\frac{\sin x}{\cos x}\right)}{dx} = \frac{d\left(\sin x (\cos x)^{-1}\right)}{dx}$

$= \sin x (-1)(\cos x)^{-2}(-\sin x) + \cos x (\cos x)^{-1}$

$= \frac{+\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ //

Example 3.10 Let $y = \sin(3x^2+8)$. Find $\frac{dy}{dx}$.

Solution: $\frac{dy}{dx} = \frac{d \sin(3x^2+8)}{dx} = \cos(3x^2+8) \cdot 6x$

$= 6x \cos(3x^2+8)$.

Example 3.17 Let $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Find

$\frac{d \cosh x}{dx}$, $\frac{d^2(\cosh x)}{dx^2}$, $\frac{d(\sinh x)}{dx}$, $\frac{d^2}{dx^2}(\sinh x)$.

Solution $\frac{d \cosh x}{dx} = \frac{d\, \frac{1}{2}(e^x + e^{-x})}{dx} = \frac{1}{2}\left(\frac{de^x}{dx} + \frac{de^{-x}}{dx}\right)$

$= \frac{1}{2}\left(e^x + (-1)e^{-x}\right) = \sinh x$.

$$\frac{d \sinh x}{dx} = \frac{d\left(\frac{1}{2}(e^{x}-e^{-x})\right)}{dx} = \frac{1}{2}\left(\frac{de^{x}}{dx} - \frac{de^{-x}}{dx}\right)$$

$$= \frac{1}{2}\left(e^{x}-(-1)e^{-x}\right) = \frac{1}{2}\left(e^{x}+e^{-x}\right) = \cosh x.$$

Then

$$\frac{d^2}{dx^2}(\cosh x) = \frac{d}{dx}(\sinh x) = \cosh x \quad \text{and}$$

$$\frac{d^2}{dx^2}(\sinh x) = \frac{d}{dx}(\cosh x) = \sinh x.$$