Calculus I Lecture 15

Example 2.38 Simplify

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2$$

Solution: $\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{v}\cdot(\vec{v}-\vec{u}) + \vec{u}(\vec{v}-\vec{u})$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{u}$$

$$= \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{v} = 2\,\vec{u}\cdot\vec{v}$$

So $\vec{u}\cdot\vec{v} = \frac{1}{2}\left(\|u\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2\right)$. //

Example 2.40 Find the angle θ between

$\vec{u} = (1, -2, 0)$ and $\vec{v} = (3, 1, -2)$

solution: Since $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ then

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = \arccos\left(\frac{(1,-2,0)\cdot(3,1,-2)}{\|(1,-2,0)\|\|(3,1,-2)\|}\right)$$

$$= \arccos\left(\frac{3\cdot 1 + (-2)\cdot 1 + 0\cdot 2}{\sqrt{1^2+2^2+0^2}\sqrt{3^2+1^2+2^2}}\right)$$

$$= \arccos\left(\frac{1}{\sqrt{5}\sqrt{14}}\right) = \arccos\left(\frac{1}{\sqrt{70}}\right). //$$

<u>Example 2.41</u> Let $\vec{u}=(3,1,-2)$ and $\vec{v}=(1,0,5)$. Find the parallel and perpendicular projection of $\vec{v}$ onto $\vec{u}$.

<u>Solution</u> We want $\vec{w}$ and $\vec{z}$ so that

$\vec{w}=c\vec{u}$ ($\vec{w}$ is parallel to $\vec{u}$)

$\vec{z}\cdot\vec{u}=0$ ($\vec{z}$ is perpendicular to $\vec{u}$)

and

$\vec{w}+\vec{z}=\vec{v}$

Let $\vec{w}=(w_1,w_2,w_3)$ and $\vec{z}=(z_1,z_2,z_3)$

Then $(w_1,w_2,w_3)=c\vec{u}=c(3,1,-2)=(3c,c,-2c)$

and $0=(z_1,z_2,z_3)\cdot(3,1,-2)=3z_1+z_2-2z_3$

and $(3c,c,-2c)+(z_1,z_2,z_3)=(1,0,5)$.

So
$$3c+z_1=1$$
$$c+z_2=0$$
$$-2c+z_3=5$$
$$3z_1+z_2-2z_3=0.$$

So
$$z_1=1-3c$$
$$z_2=-c$$
$$z_3=5+2c$$
$$3(1-3c)+(-c)-2(5+2c)=0.$$

So $3-9c-c-10-4c=0$. So $-7-14c=0$.

So $c=\frac{-1}{2}$

$z_1=1-\left(\frac{-3}{2}\right)=\frac{5}{2}$

$z_2=+\frac{1}{2}$

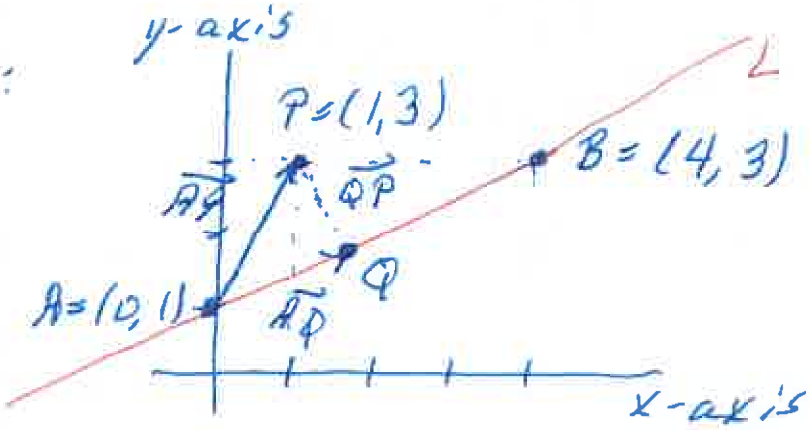
$z_3=5+2\left(\frac{-1}{2}\right)=4$.

So $\vec{w}=\frac{-1}{2}\vec{u}=\frac{-1}{2}(3,1,-2)=\left(\frac{-3}{2},\frac{-1}{2},1\right)$

$\vec{z}=(z_1,z_2,z_3)=\left(\frac{5}{2},\frac{1}{2},4\right)$.

Example 2.42 Let $P=(1,3)$ and let L be the line passing through $A=(0,1)$ and $B=(4,3)$. Find the closest point Q on the line L to P and hence find the distance from P to L.

Solution:



$$\vec{AP} = P - A = (1,3)-(0,1) = (1,2)$$

$$\vec{AQ} + \vec{QP} = \vec{AP}$$

$$\vec{AQ} = c\,\vec{AB} = c(B-A) = c((4,3)-(0,1))$$
$$= c(4,2) = (4c, 2c)$$

$$\vec{AB} \cdot \vec{QP} = 0.$$

Let $Q=(q_1, q_2)$. Then

$$\vec{QP} = P - Q = (1,3)-(q_1,q_2) = (1-q_1, 3-q_2),$$

$$0 = \vec{AB}\cdot\vec{QP} = (4,2)\cdot(1-q_1, 3-q_2)$$
$$= 4-4q_1+6-3q_2 = 10-4q_1-3q_2$$

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and $(1,2) = \overrightarrow{AP} = \overrightarrow{AQ} + \overrightarrow{QP}$

$$= (4c, 2c) + (1-q_1, 3-q_2)$$
$$= (4c+1-q_1, 2c+3-q_2).$$

So
$4c+1-q_1 = 1$, so $q_1 = 4c$
$2c+3-q_2 = 2$, $q_2 = 2c+1$
$10-4q_1-3q_2=0$. $10-4(4c)-3(2c+1)=0$.

So $10-16c-6c-3=0$. So $22c=7$.

So $c = \frac{7}{22}$

$q_1 = 4c = 4\cdot\frac{7}{22} = \frac{28}{22} = \frac{14}{11}$

$q_2 = 2c+1 = 2\cdot\frac{7}{22}+1 = \frac{7}{11}+1 = \frac{18}{11}$

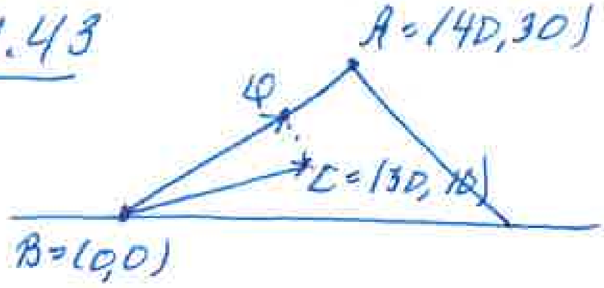
So $Q = \left(\frac{14}{11}, \frac{18}{11}\right)$ and

distance from P to L $= \|\overrightarrow{QP}\| = \|(1-q_1, 3-q_2)\|$

$$= \left\|\left(1-\frac{14}{11}, 3-\frac{18}{11}\right)\right\| = \left\|\left(\frac{-3}{11}, \frac{15}{11}\right)\right\|$$

$$= \sqrt{\left(\frac{3}{11}\right)^2+\left(\frac{15}{11}\right)^2} = \frac{1}{11}\sqrt{9+225} = \frac{\sqrt{234}}{11}.$$

Example 2.43



Where should Lara Croft start digging to reach the chamber at C? How long will the shaft be?

Solution $\overrightarrow{BQ} = c\overrightarrow{BA} = c(A-B) = c((40,30)-(0,0))$

$$= c(40,30) = (40c, 30c)$$

$\overrightarrow{QC}\cdot\overrightarrow{BA} = 0$, so, if $Q=(q_1,q_2)$ then

$$0 = \overrightarrow{QC}\cdot\overrightarrow{BA} = (C-Q)\cdot(A-B) = ((30,10)-(q_1,q_2))\cdot(40,30)$$

$$= (30-q_1, 10-q_2)\cdot(40,30)$$

$$= 120-40q_1+300-30q_2$$

Then $\overrightarrow{BC} = (30,10)$ and

$$(30,10) = \overrightarrow{BC} = \overrightarrow{BQ}+\overrightarrow{QC} = (40c,30c) + (30-q_1, 10-q_2)$$

$$= (40c+30-q_1, 30c+10-q_2).$$

So $40c+30-q_1 = 30$, so $q_1 = 40c$

$30c+10-q_2 = 10$, $q_2 = 30c$

$420-40q_1-30q_2 = 0$. $420-40\cdot 40c - 30\cdot 30c = 0$.

So $420 = 2500c$ and $c = \frac{42}{250}$, $q_1 = \frac{1680}{250}$, $q_2 = \frac{1260}{250}$.

The length of the shaft is

$$\|\vec{QC}\| = \|(30-q_1, 10-q_2)\| = \left\|\left(30 - \frac{168}{25}, 10 - \frac{126}{25}\right)\right\|$$

$$= \left\|\left(\frac{750-168}{25}, \frac{250-126}{25}\right)\right\|$$

$$= \frac{1}{25}\sqrt{(582)^2+(124)^2}.$$