Calculus I Lecture 10 19.08.2019

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Example 2.1 and 2.2

(a) Is $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = \dfrac{x+1}{x^2+4}$ a function?

(b) Is $g: \mathbb{C} \to \mathbb{C}$ given by $g(x) = \dfrac{x+1}{x^2+4}$ a function?

Solution: The expression $\dfrac{x+1}{x^2+4}$ is not defined when x^2+4. This is when $x^2 = -4$ so that $x = \sqrt{-4} = \sqrt{-1}\sqrt{4} = \pm 2i$.

So $\dfrac{x+1}{x^2+4}$ is not defined when $x = 2i$ or $x = -2i$.

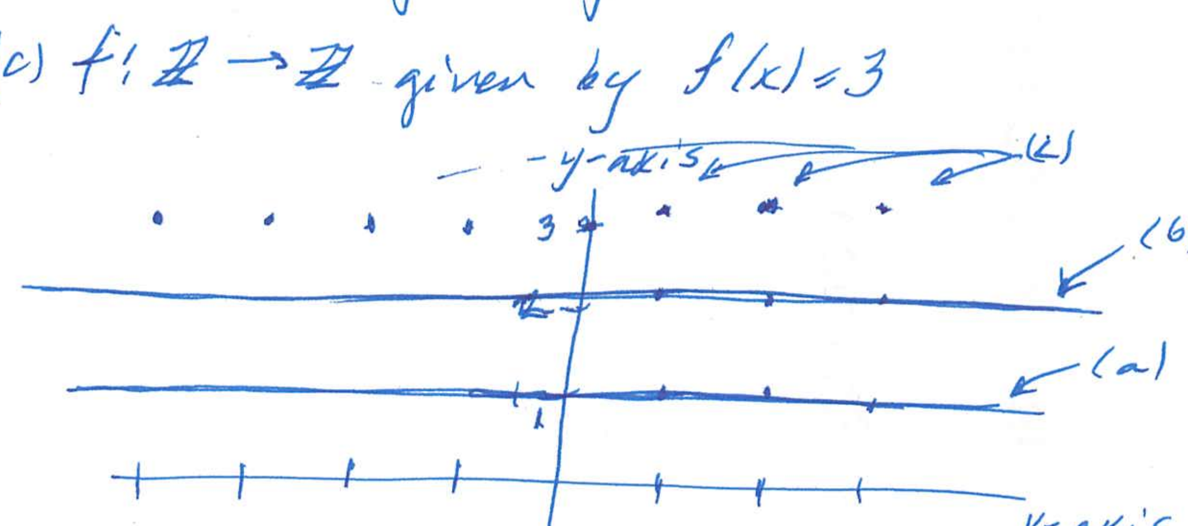
So $g: \mathbb{R} \to \mathbb{R}$ is a function and $g: \mathbb{C} \to \mathbb{C}$ is not. //

Example 2.3 Sketch the graphs of

(a) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$,

(b) $f: \mathbb{R} \to \mathbb{Z}$ given by $f(x) = 2$

(c) $f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3$

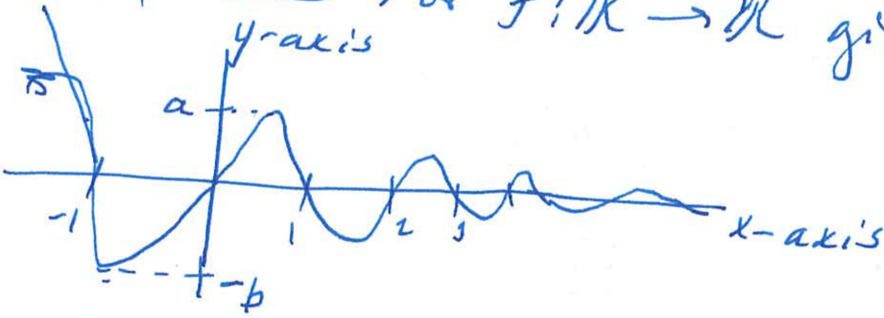


Example 2.4 $\alpha : \{words\} \longrightarrow \{alphabet\}$

$w \longmapsto$ first letter

So $\alpha(maths) = m$, $\alpha(fun) = f$, $\alpha(feather) = f$.

Example 2.5 For $f: \mathbb{R} \to \mathbb{R}$ given by the graph



$f([0,1]) = [0, a]$

$f((-\infty, -1]) =$ uncertain but probably $[0, \infty)$

$f([-1, 2]) = [-b, a]$.

$\mathrm{range}(f) = \mathrm{im}(f) = [-b, \infty)$.

Example 2.6 Show that the function

$\log: (0, \infty) \to \mathbb{R}$ is increasing and injective.

Solution: To show: If $x < y$ then $\log(x) < \log(y)$

Using that e^x is increasing,

To show: If $x < y$ then $e^{\log(x)} < e^{\log(y)}$

To show: If $x < y$ then $x < y$.

Yes this is true.

To show: If $x_1, x_2 \in [0,\infty)$ and $x_1 \neq x_2$

then $\log(x_1) \neq \log(x_2)$

Case 1 $x_1 < x_2$.

Then $\log(x_1) < \log(x_2)$

So $\log(x_1) \neq \log(x_2)$

Case 2 $x_2 < x_1$

Then $\log(x_2) < \log(x_1)$

So $\log(x_2) \neq \log(x_1)$. //

Example 2.7 Find the largest intervals on which $\begin{aligned} f: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$ is injective.

Solution Since $f(x) = f(y)$

if and only if $x^2 = y^2$

if and only if $x = \pm y$

then

$[0,\infty)$ and $(-\infty, 0]$ are the largest intervals on which f is injective. //

Example 2.8 Prove that $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ given by $g(x) = \frac{2x-7}{1-x}$ is injective.

Proof To show: If $x_1, x_2 \in \mathbb{R} \setminus \{1\}$ and $g(x_1) = g(x_2)$ then $x_1 = x_2$.

Assume $x_1, x_2 \in \mathbb{R} \setminus \{1\}$ and $g(x_1) = g(x_2)$.

So $x_1 \neq 1$, $x_2 \neq 1$ and $\frac{2x_1 - 7}{1-x_1} = \frac{2x_2-7}{1-x_2}$.

To show: $x_1 = x_2$.

Since $\frac{2x_1-7}{1-x_1} = \frac{2x_2-7}{1-x_2}$ then $(2x_1-7)(1-x_2)$
$= (2x_2-7)(1-x_1)$.

So $2x_1 - 2x_1x_2 - 7 + 7x_2 = 2x_2 - 2x_2x_1 - 7 + 7x_1$

So $2x_1 + 7x_2 = 2x_2 + 7x_1$

So $5x_2 = 5x_1$

So $x_2 = x_1$

So g is injective //.

Example 2.9 Give a table definition for

$f: \{0,1,2\} \to \{0,1,2\}$ given by

$$f(x) = \begin{cases} x+1, & \text{if } x \neq 2, \\ 0, & \text{if } x = 2. \end{cases}$$

Solution

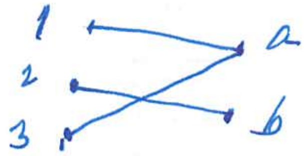
$f(0) = 0+1 = 1,$

$f(1) = 1+1 = 2,$

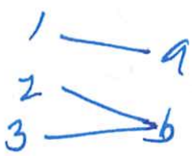
$f(2) = 0.$

Example 2.10 Give definitions of all surjective functions $f: \{1,2,3\} \to \{a,b\}$

Solution:

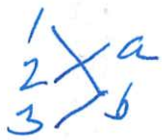


$f(1) = a$
$f(2) = b$
$f(3) = a$

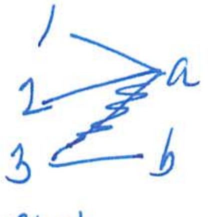


$f(1) = a$
$f(2) = b$
$f(3) = b$

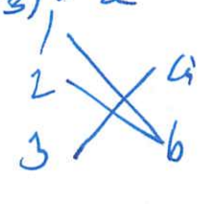
$f(1) = b$
$f(2) = a$
$f(3) = a$



$f(1) = b$
$f(2) = a$
$f(3) = b$



$f(1) = a$
$f(2) = a$
$f(3) = b$



$f(1) = b$
$f(2) = b$
$f(3) = a$