Vector Calculus Lecture 9

10.08.2018
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Space curves in $\mathbb{R}^3$

$$\text{curve} = \vec{c}(t) = (x(t), y(t), z(t))$$

$$\text{velocity} = \frac{d\vec{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\text{speed} = \left|\frac{d\vec{c}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{ds}{dt}$$

$$\text{arclength from } t=a \text{ to } t=b = \int_{t=a}^{t=b} ds = \int_{t=a}^{t=b} \frac{ds}{dt}\,dt = \int_{t=a}^{t=b} \left|\frac{d\vec{c}}{dt}\right| dt$$

tangent line to $\vec{c}$ at $t=a$:

$$(x,y,z) = \vec{c}(a) + (t-a)\left(\left.\frac{d\vec{c}}{dt}\right|_{t=a}\right)$$

$$\text{unit tangent vector} = \vec{T}(t) = \frac{\frac{d\vec{c}}{dt}}{\left|\frac{d\vec{c}}{dt}\right|}$$

$$\text{principal normal vector} = \vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

$$\text{binormal vector} = \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\text{curvature} = \kappa(t) = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{d\vec{c}}{dt}\right|} \quad \text{and} \quad \frac{\frac{d\vec{B}}{dt}}{\left|\frac{d\vec{c}}{dt}\right|} = -\tau(t)\vec{N}$$

torsion.

§2.1 Example 3 Let

$$\vec{c}(t) = (3t+2, t^2-7, t-t^2)$$

Determine the velocity, speed, acceleration and equation of the tangent line at $t=1$.

Solution

velocity $= \frac{d\vec{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (3, 2t, 1-2t)$

speed $= \left|\frac{d\vec{c}}{dt}\right| = \sqrt{3^2+(2t)^2+(1-2t)^2}$

$= \sqrt{9+4t^2+1-4t+4t^2}$

$= \sqrt{10-4t+8t^2}$

acceleration $= \frac{d^2\vec{c}}{dt^2} = (0, 2, -2)$

Then, at $t=1$,

$$\vec{c}(1) = (3+2, 1-7, 1-1) = (5, -6, 0),$$

$$\left.\frac{d\vec{c}}{dt}\right|_{t=1} = (3, 2, 1-2) = (3, 2, -1),$$

$$\left|\frac{d\vec{c}}{dt}\right|_{t=1} = \sqrt{10-4+8} = \sqrt{14},$$

$$\left.\frac{d^2c}{dt^2}\right|_{t=1} = (0, 2, -2), \quad \text{and}$$

$$(x, y, z) = \vec{c}(1) + (t-1)\frac{d\vec{c}}{dt} = (5, -6, 0) + (t-1)(3, 2, -1)$$

is the equation of the tangent line at $t=1$

§2.1 Example 5 Simplify

$$\frac{d}{dt}\left((u'' \times u') \cdot u\right)$$

Solution:

$$\frac{d}{dt}\left((u''\times u')\cdot u\right) = \frac{d}{dt}(u''\times u')\cdot u + (u''\times u')\cdot \frac{du}{dt}$$

$$= \frac{d}{dt}(u''\times u')\cdot u + (u''\times u')\cdot u'$$

$$= \frac{d}{dt}(u''\times u')\cdot u + 0$$

$$= \left(\frac{du''}{dt}\times u' + u''\times \frac{du'}{dt}\right)\cdot u$$

$$= (u'''\times u' + u''\times u'')\cdot u = (u'''\times u' + 0)\cdot u$$

$$= (u'''\times u')\cdot u.$$

§2.1 Example 6 Let

$$c(t) = (2t, t^2, \log t) \quad \text{for} \quad t \in \mathbb{R}_{[1,2]}.$$

Find the arclength.

Solution: arclength of $\vec{c}$ from $t=1$ to $t=2$ $= \displaystyle\int_{t=1}^{t=2} \left|\frac{d\vec{c}}{dt}\right| dt.$

$$\frac{d\vec{c}}{dt} = \left(2, 2t, \frac{1}{t}\right) \quad \text{and}$$

$$\left|\frac{d\vec{c}}{dt}\right| = \sqrt{2^2 + (2t)^2 + \frac{1}{t^2}} = \sqrt{4 + 4t^2 + \frac{1}{t^2}}$$

$$= \sqrt{\frac{4t^2 + 4t^4 + 1}{t^2}} = \sqrt{\frac{(1+2t^2)(1+2t^2)}{t^2}}$$

$$= \frac{1+2t^2}{t}$$

So

$$\int_{t=1}^{t=2} \left|\frac{d\vec{c}}{dt}\right| dt = \int_{t=1}^{t=2} \left(\frac{1+2t^2}{t}\right) dt = \int_{t=1}^{t=2} \left(\frac{1}{t} + 2t\right) dt$$

$$= (\log t + t^2)\Big|_{t=1}^{t=2} = (\log 2 + 4) - (\log 1 + 1)$$

$$= \log 2 + 3.$$

§2.1 Example 7 Let

$$\vec{c}(t) = (5\cos 3t, 6t, 5\sin 3t)$$

Find $\vec{T}(t)$, $\vec{N}(t)$, $\vec{B}(t)$, $\kappa(t)$ and $\tau(t)$.

Solution

$$\text{velocity} = \frac{d\vec{c}}{dt} = (5(-\sin 3t)\cdot 3,\ 6,\ 5(\cos 3t)\cdot 3)$$

$$= (-15\sin 3t,\ 6,\ 15\cos 3t)$$

$$\text{speed} = \left|\frac{d\vec{c}}{dt}\right| = \sqrt{(-15\sin 3t)^2 + 6^2 + (15\cos 3t)^2}$$

$$= \sqrt{15^2\sin^2 3t + 6^2 + 15^2\cos^2 3t}$$

$$= \sqrt{15^2 + 6^2} = \sqrt{3^2\cdot 5^2 + 3^2\cdot 2^2} = 3\sqrt{5^2+2^2}$$

$$= 3\sqrt{29}$$

$$\vec{T}(t) = \frac{\frac{d\vec{c}}{dt}}{\left|\frac{d\vec{c}}{dt}\right|} = \frac{1}{3\sqrt{29}}(-15\sin 3t,\ 6,\ 15\cos 3t)$$

$$= \frac{1}{\sqrt{29}}(-5\sin 3t,\ 2,\ 5\cos 3t)$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{29}}(-5(\cos 3t)\cdot 3,\ 0,\ 5(-\sin 3t)\cdot 3)$$

$$= \frac{1}{\sqrt{29}}(-15\cos 3t,\ 0,\ -15\sin 3t)$$

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$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(\frac{(-15\cos 3t)}{\sqrt{29}}\right)^2 + 0^2 + \left(\frac{(-15\sin 3t)}{\sqrt{29}}\right)^2}$$

$$= \sqrt{\frac{15^2}{29}\cos^2 3t + \frac{15^2}{29}\sin^2 3t}$$

$$= \sqrt{\frac{15^2}{29}} = \frac{15}{\sqrt{29}}$$

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \frac{\left(\frac{-15}{\sqrt{29}}\cos 3t,\ 0,\ \frac{-15}{\sqrt{29}}\sin 3t\right)}{\frac{15}{\sqrt{29}}}$$

$$= (-\cos 3t,\ 0,\ -\sin 3t)$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{5}{\sqrt{29}}\sin 3t & \frac{2}{\sqrt{29}} & \frac{5}{\sqrt{29}}\cos 3t \\ -\cos 3t & 0 & -\sin 3t \end{vmatrix}$$

$$= \hat{i}\left(\frac{-2}{\sqrt{29}}\sin 3t - 0\right) - \hat{j}\left(\frac{5}{\sqrt{29}}\sin^2 3t + \frac{5}{\sqrt{29}}\cos^2 3t\right)$$

$$+ \hat{k}\left(0 + \frac{2}{\sqrt{29}}\cos 3t\right)$$

$$= -\frac{2}{\sqrt{29}}\sin 3t\,\hat{i} - \frac{5}{\sqrt{29}}\hat{j} + \frac{2}{\sqrt{29}}\cos 3t\,\hat{k}$$

$$= \frac{1}{\sqrt{29}}(-2\sin 3t, -5, 2\cos 3t).$$

So

$$\frac{\frac{d\vec{B}}{dt}}{\left|\frac{d\vec{c}}{dt}\right|} = \frac{1}{3\sqrt{29}}\left(\frac{1}{\sqrt{29}}(-2)(\cos 3t)\cdot 3, 0, \frac{1}{\sqrt{29}}2(-\sin 3t)\cdot 3\right)$$

$$= \frac{1}{29}(-2\cos 3t, 0, -2\sin 3t)$$

$$= \frac{+2}{29}(-\cos 3t, 0, -\sin 3t) = \frac{2}{29}\vec{N}(t),$$

giving $\tau(t) = \frac{\cancel{2}}{29} = -\frac{2}{29}$