§1.5 Example Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(x+2y)}{x+2y}$.

Solution: Let $z = x+2y$

As $(x,y)\to(0,0)$ then $z\to 0$.

Using the Taylor series for $\sin z$, which is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots$$

then

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+2y)}{x+2y} = \lim_{z\to 0} \frac{\sin z}{z}$$

$$= \lim_{z\to 0} \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots}{z}$$

$$= \lim_{z\to 0} 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

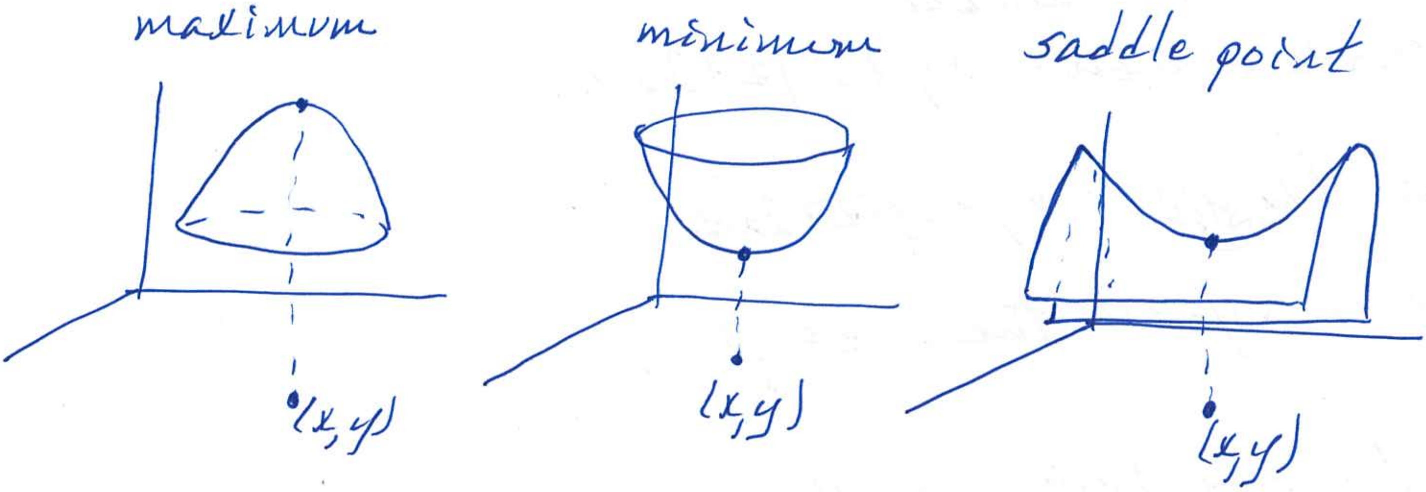
$$= 1 - 0 + 0 - \cdots = 1.$$

Finding extrema: maxima, minima, ...

Two steps:

Step 1: Find the critical points

Step 2: Determine whether the critical point is a



The gradient of f

If $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ then

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy.$$

If $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ then

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Critical points of f

Case 0: No constraints

Critical points are when $\vec{\nabla} f = 0$.

Case 1: One constraint: $g_1 = 0$.

Critical points are when $\vec{\nabla} f = \lambda_1 \vec{\nabla} g_1$

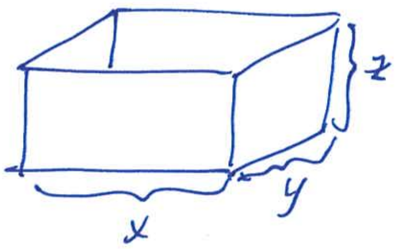
Case 2: Two constraints: $g_1 = 0$ and $g_2 = 0$

Critical points are when

$$\vec{\nabla} f = \lambda_1 \vec{\nabla} g_1 + \lambda_2 \vec{\nabla} g_2 .$$

<u>§1.6 Example 1</u> A rectangular box must have volume $4\,m^3$ and minimum surface area. Find the dimensions of the box.

<u>Solution:</u>



$\text{Volume} = xyz = 4$

We want to minimize

$$f = \text{Surface area} = xy + 2yz + 2xz$$

Since $xyz = 4$ then $z = \frac{4}{xy}$ and

$$f = xy + 2y \cdot \frac{4}{xy} + 2x \cdot \frac{4}{xy} = xy + \frac{8}{x} + \frac{8}{y}$$

The critical points are when

$$(0,0) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(y - \frac{8}{x^2}, x - \frac{8}{y^2}\right)$$

Then $y - \frac{8}{x^2} = 0$ gives $y = \frac{8}{x^2}$.

then $x - \frac{8}{y^2} = 0$ gives $0 = x - \frac{8}{(8/x^2)^2} = x - \frac{8}{8^2}x^4$

So $$0 = \frac{x}{8}(8 - x^3).$$

So $x = 0$ or $x = 2$.

Since $xyz = 4$ then $x = 0$ is not possible.

So

$x = 0$ and $y = \frac{8}{x^2} = \frac{8}{4} = 2$ and $z = \frac{4}{xy} = \frac{4}{2 \cdot 2} = \frac{4}{4} = 1$.

So the only critical point is $(2,2,1)$.

The Hessian matrix is

$$\underline{\underline{H}}f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y} \end{pmatrix} = \begin{pmatrix} (-8)(-2)x^{-3} & 1 \\ 1 & (-8)(-2)y^{-3} \end{pmatrix} = \begin{pmatrix} \frac{16}{x^3} & 1 \\ 1 & \frac{16}{y^3} \end{pmatrix}.$$

Then

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(2,2)} = \left.\frac{16}{x^3}\right|_{(x,y)=(2,2)} = \frac{16}{8} = 2 \in \mathbb{R}_{>0}$$

$$\det\left(\underline{\underline{H}}f\big|_{(x,y)=(2,2)}\right) = \det\begin{pmatrix} \frac{16}{8} & 1 \\ 1 & \frac{16}{8} \end{pmatrix} = 2 \cdot 2 - 1 \cdot 1 = 4 - 1 = 3 \in \mathbb{R}_{>0}$$

So $(x,y) = (2,2)$ is a minimum of f.

So $(x,y,z) = (2,2,1)$ minimizes surface area.

§1.7 Example 4 Minimise $f(x,y,z) = xy + 2xz + 2yz$ subject to $xyz = 4$.

Solution: The constraint is $g(x,y,z) = 0$ where $g(x,y,z) = xyz - 4$.

Using Lagrange multipliers, the critical points are when

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \text{ which is}$$

$$\left(\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y} \ \frac{\partial f}{\partial z}\right) = \lambda \left(\frac{\partial g}{\partial x} \ \frac{\partial g}{\partial y} \ \frac{\partial g}{\partial z}\right), \text{ which is}$$

$$(y+2z,\ x+2z,\ 2x+2y) = \lambda(yz,\ xz,\ xy).$$

If
$$y + 2z = \lambda yz$$
$$x + 2z = \lambda xz$$
$$2x + 2y = \lambda xy$$
$$xyz = 4$$
then $y - x = \lambda(y-x)z$ so that $(y-x)(1-\lambda z) = 0$ giving $y = x$ or $z = \frac{1}{\lambda}$.

Since $xyz = 4$ then $x \neq 0$ and $y \neq 0$ and $z \neq 0$.

Case 1: $z = \frac{1}{\lambda}$: Then $y + 2z = \lambda yz = \lambda y \frac{1}{\lambda} = y$.

So $2z = 0$. So $z = 0$ but this contradicts $z \neq 0$.

Case 2: $x=y$. Then $4 = xyz = y^2 z$ so that $z = \frac{4}{y^2}$.

Then
$$\begin{aligned} y + 2z &= \lambda yz \\ 2y + 2y &= \lambda y^2 \end{aligned} \quad \text{gives} \quad \begin{aligned} y + \frac{8}{y^2} &= \lambda y \cdot \frac{4}{y^2} \\ 4y &= \lambda y^2 \end{aligned}$$

So $\lambda y = 4$ and $y + \frac{8}{y^2} = \frac{4\cdot 4}{y^2} = \frac{16}{y^2}$.

So $y^3 + 8 = 16$. So $y^3 = 8$. So $y = 2$.

So $x = 2$ and $z = \frac{4}{2^2} = \frac{4}{4} = 1$

So the critical point is at $(x,y,z) = (2,2,1)$.

This either a maximum or a minimum since xyz is a closed bounded constraint.

Since
$$f(1,1,4) = 1 + 2\cdot 4 + 2\cdot 4 = 17 \quad \text{and}$$
$$f(2,2,1) = 4 + 2\cdot 2 + 2\cdot 2 = 12$$
then $(x,y,z) = (2,2,1)$ must be a minimum and not a maximum.