Vector Calculus Lecture 35

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35.6 Example 1 (Gravitation)

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2+y^2+z^2}$. Let

$$\vec{F} = \frac{1}{r^3}\vec{r} = \frac{r}{r^3}\hat{r} = \frac{1}{r^2}\hat{r}$$

Let Ω be a region in $\mathbb{R}^3$. Assume $0 \notin \partial\Omega$.

Show that

$$\iint_{\partial\Omega} \vec{F}\cdot d\vec{S} = \begin{cases} 0, & \text{if } 0 \notin \Omega, \\ 4\pi, & \text{if } 0 \in \Omega. \end{cases}$$

Solution:

Case 1: $0 \notin \Omega$. Then $\vec{F}$ is C^1 in Ω.

Using the Divergence theorem.

$$\iint_{\partial\Omega} \vec{F}\cdot d\vec{S} = \iiint_{\Omega} (\vec{\nabla}\cdot\vec{F})\, dV$$

and

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot\left(\frac{1}{r^3}\vec{r}\right) = \vec{\nabla}\cdot\left(\frac{1}{r^2}\hat{r}\right)$$

$$= \frac{1}{h_r h_\varphi h_\theta}\left(\frac{\partial}{\partial r}(h_\varphi h_\theta F_r) + \frac{\partial}{\partial\varphi}(h_r h_\theta F_\varphi) + \frac{\partial}{\partial\theta}(h_r h_\varphi F_\theta)\right)$$

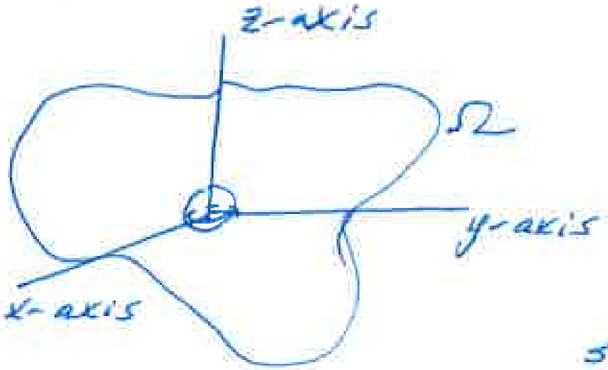
$$= \frac{1}{1\cdot r\cdot r\sin\theta}\left(\frac{\partial}{\partial r}\left(r r\sin\theta \frac{1}{r^2}\right) + \frac{\partial}{\partial\varphi}0 + \frac{\partial}{\partial\theta}0\right)$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(\sin\theta) = 0.$$

So

$$\iint_{\partial\Omega} \vec{F}\cdot d\vec{S} = 0.$$

<u>Case 2</u> $O \in \Omega$. In this case $\vec{F}$ is <u>not</u> C^1 in Ω.



V is the region between $\partial\Omega$ and the inner sphere ~~circle~~ of radius a.

Using the Divergence theorem (note that $\vec{F}$ is C^1 on V since $O \notin V$) then

$$\iint_{\partial\Omega} \vec{F}\cdot dS = \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV - \iint_{\substack{\text{inner}\\\text{boundary}}} \vec{F}\cdot d\vec{S}$$

$$= 0 - \iint_{\substack{\text{inner}\\\text{boundary}}} \vec{F}\cdot\hat{n}\,dS = -\iint_{\substack{\text{inner}\\\text{boundary}}} \vec{F}\cdot(-\hat{r})\,dS$$

$$= +\iint_{\substack{\text{inner}\\\text{boundary}}} \frac{1}{r^2}\hat{r}\cdot\hat{r}\,dS = +\iint_{\substack{\text{inner}\\\text{boundary}}} \frac{1}{r^2}|\hat{r}|^2\,dS$$

$$= +\iint_{\substack{\text{inner}\\\text{boundary}}} \frac{1}{a^2}1^2\,dS = +\frac{1}{a^2}\left(\begin{array}{l}\text{surface area of}\\ \text{inner sphere of radius } a\end{array}\right)$$

$$= \frac{1}{a^2}(4\pi a^2) = 4\pi.$$

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35.6 Example 2 (Fluid flow)

$\rho(\vec{r},t)$ = fluid density

$v(\vec{r},t)$ = fluid velocity

$\vec{J}(r,t)$ = fluid current.

Assume

$\vec{J} = \rho\vec{v}$. Show that $\dfrac{\partial\vec{\rho}}{\partial t} = -\vec{\nabla}\cdot\vec{J}$.

Solution Let V be a region.

$$M = \text{Mass of fluid in } V = \iiint_V \vec{\rho}\, dV$$

Then

$$\iiint_V \frac{\partial\vec{\rho}}{\partial t} dV = \frac{\partial}{\partial t}\iiint_V \vec{\rho}\, dV = \frac{\partial M}{\partial t}$$

$$= -\text{ mass of fluid leaving } V = -\text{ flux of current out of } V$$

$$= -\iint_{\partial V} \vec{J}\cdot d\vec{S} = -\iiint_V \vec{\nabla}\cdot\vec{J}\, dV,$$

by the Divergence theorem.

So $\dfrac{\partial\vec{\rho}}{\partial t} = -\vec{\nabla}\cdot\vec{J}$.

Note: $-\vec{\nabla}\cdot\vec{J} = -\vec{\nabla}\cdot(\rho\vec{v}) = -(\vec{\nabla}\rho)\cdot v - \rho\,\vec{\nabla}\cdot\vec{v}$

$$= -(\vec{\nabla}\rho)\cdot v - 0 = -(\vec{\nabla}\rho)\cdot\vec{v}.$$

35.6 Example 3 (Electomagnetic fields) A. Ram

$\vec{E}$ = electric field

$\vec{B}$ = magnetic field

ρ = charge density

$\vec{J}$ = current.

Maxwell's equations

(a) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

(b) $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$

(c) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t}$

(d) $\vec{\nabla}\cdot\vec{B} = 0$

(1) Assume $\vec{B}$ is constant in time.
Show that $\vec{E}$ is a gradient field.

Solution $0 = \dfrac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$, by equation (a).
So there exists φ such that $\vec{E} = \vec{\nabla}\varphi$.

(2) Assume $\vec{E}$ is constant in time.
Show that if $\vec{J}\neq 0$ then $\vec{B}\neq 0$.

Solution
$0 = \dfrac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} - \mu_0\vec{J}$.

So, if $\vec{J}\neq 0$ then $\vec{\nabla}\times\vec{B}\neq 0$. So $\vec{B}\neq 0$.

(3) Assume $\vec{J} = 0$ and $\rho = 0$ (no currents, no charges.

Then

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}.$$

Show that

$$\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \qquad \left(\begin{array}{l}\text{wave equation} \\ \text{in a vacuum}\end{array}\right)$$

<u>Solution</u> From identity 15 on Formula sheet 1,

$$\nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$

$$= \vec{\nabla} 0 - \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = \frac{\partial}{\partial t}\left(\varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right) = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t}.$$

<u>Note</u>: An electromagnetic wave travels with speed

$$\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 300{,}000 \text{ km/sec}$$

as predicted by Maxwell 20 years before experimentally observed by Hertz.