Vector Calculus Lecture 32

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Consider a change of variables

$$u = u(x,y,z) \qquad x = x(u,v,w)$$
$$v = v(x,y,z) \qquad y = y(u,v,w)$$
$$w = w(x,y,z) \qquad z = z(u,v,w)$$

Then

$$\vec{u} = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k}$$

$$\vec{v} = \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath} + \frac{\partial z}{\partial v}\hat{k}$$

$$\vec{w} = \frac{\partial x}{\partial w}\hat{\imath} + \frac{\partial y}{\partial w}\hat{\jmath} + \frac{\partial z}{\partial w}\hat{k}$$

and

$$h_u = |\vec{u}| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$$

$$h_v = |\vec{v}| = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}$$

$$h_w = |\vec{w}| = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2}$$

and

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{h_u}\vec{u} = \frac{1}{h_u}\frac{\partial x}{\partial u}\hat{\imath} + \frac{1}{h_u}\frac{\partial y}{\partial u}\hat{\jmath} + \frac{1}{h_u}\frac{\partial z}{\partial u}\hat{k}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{h_v}\vec{v} = \frac{1}{h_v}\frac{\partial x}{\partial v}\hat{\imath} + \frac{1}{h_v}\frac{\partial y}{\partial v}\hat{\jmath} + \frac{1}{h_v}\frac{\partial z}{\partial v}\hat{k}$$

$$\hat{w} = \frac{\vec{w}}{|\vec{w}|} = \frac{1}{h_w}\vec{w} = \frac{1}{h_w}\frac{\partial x}{\partial w}\hat{\imath} + \frac{1}{h_w}\frac{\partial y}{\partial w}\hat{\jmath} + \frac{1}{h_w}\frac{\partial z}{\partial w}\hat{k}.$$

Cylindrical coordinates

$$x = \rho\cos\varphi \qquad \rho = \sqrt{x^2+y^2}$$
$$y = \rho\sin\varphi \qquad \varphi = \arctan(y/x)$$
$$z = z \qquad z = z$$

Then

$$\vec{\rho} = \frac{\partial x}{\partial \rho}\hat{\imath} + \frac{\partial y}{\partial \rho}\hat{\jmath} + \frac{\partial z}{\partial \rho}\hat{k} = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath} + 0\hat{k}$$

$$\vec{\varphi} = \frac{\partial x}{\partial \varphi}\hat{\imath} + \frac{\partial y}{\partial \varphi}\hat{\jmath} + \frac{\partial z}{\partial \varphi}\hat{k} = -\rho\sin\varphi\,\hat{\imath} + \rho\cos\varphi\,\hat{\jmath} + 0\hat{k}$$

$$\vec{z} = \frac{\partial x}{\partial z}\hat{\imath} + \frac{\partial y}{\partial z}\hat{\jmath} + \frac{\partial z}{\partial z}\hat{k} = 0\,\hat{\imath} + 0\hat{\jmath} + 1\cdot\hat{k}.$$

Then

$$h_\rho = |\vec{\rho}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

$$h_\varphi = |\vec{\varphi}| = \sqrt{\rho^2\sin^2\varphi + \rho^2\cos^2\varphi} = \sqrt{\rho^2} = \rho$$

$$h_z = |\vec{z}| = \sqrt{1^2} = 1.$$

So

$$\hat{\rho} = \frac{1}{h_\rho}\vec{\rho} = \frac{1}{1}\vec{\rho} = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}$$

$$\hat{\varphi} = \frac{1}{h_\varphi}\vec{\varphi} = \frac{1}{\rho}(-\rho\sin\varphi\,\hat{\imath} + \rho\cos\varphi\,\hat{\jmath}) = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

$$\hat{z} = \frac{1}{h_z}\vec{z} = \frac{1}{1}\hat{k} = \hat{k}.$$

Since

$$\hat{\rho}\cdot\hat{\varphi} = -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi = 0$$

$$\hat{\rho}\cdot\hat{z} = 0+0+0 = 0$$

$$\hat{\varphi}\cdot\hat{z} = 0+0+0 = 0$$

this coordinate system is orthogonal.

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Spherical coordinates

$$x = r\sin\theta\cos\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\sin\theta\sin\varphi \qquad \varphi = \arctan(y/x)$$

$$z = r\cos\theta \qquad \theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Then

$$\vec{r} = \frac{\partial x}{\partial r}\hat{\imath} + \frac{\partial y}{\partial r}\hat{\jmath} + \frac{\partial z}{\partial r}\hat{k}$$

$$= \sin\theta\cos\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\vec{\varphi} = \frac{\partial x}{\partial\varphi}\hat{\imath} + \frac{\partial y}{\partial\varphi}\hat{\jmath} + \frac{\partial z}{\partial\varphi}\hat{k}$$

$$= -r\sin\theta\sin\varphi\,\hat{\imath} + r\sin\theta\cos\varphi\,\hat{\jmath} \overset{+\,0}{\cancel{-r\sin\theta}}\,\hat{k}$$

$$\vec{\theta} = \frac{\partial x}{\partial\theta}\hat{\imath} + \frac{\partial y}{\partial\theta}\hat{\jmath} + \frac{\partial z}{\partial\theta}\hat{k}$$

$$= r\cos\theta\cos\varphi\,\hat{\imath} + r\cos\theta\sin\varphi\,\hat{\jmath} - r\sin\theta\,k$$

and

$$h_r = |\vec{r}| = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1.$$

$$h_\varphi = |\vec{\varphi}| = \sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi}$$

$$= \sqrt{r^2\sin^2\theta} = |r\sin\theta|$$

$$h_\theta = |\vec{\theta}| = \sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r.$$

So

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{h_r} = \frac{1}{1}(\sin\theta\cos\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\theta\,\hat{k})$$

$$= \sin\theta\cos\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\hat{\varphi} = \frac{\vec{\varphi}}{|\vec{\varphi}|} = \frac{\vec{\varphi}}{h_\varphi} = \frac{1}{r\sin\theta}(-r\sin\theta\sin\varphi\,\hat{\imath} + r\sin\theta\cos\varphi\,\hat{\jmath})$$

$$= -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

$$\hat{\theta} = \frac{\vec{\theta}}{|\vec{\theta}|} = \frac{\vec{\theta}}{h_\theta} = \frac{1}{r}\left(\begin{aligned} & r\cos\theta\cos\varphi\,\hat{\imath} \\ & + r\cos\theta\sin\varphi\,\hat{\jmath} - r\sin\theta\,\hat{k} \end{aligned}\right)$$

$$= \cos\theta\cos\varphi\,\hat{\imath} + \cos\theta\sin\varphi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

Since

$$\hat{r}\cdot\hat{\varphi} = -\sin\theta\cos\varphi\sin\varphi + \sin\theta\sin\varphi\cos\varphi = 0,$$

$$\hat{r}\cdot\hat{\theta} = \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \cos\theta\sin\theta$$

$$= \sin\theta\cos\theta - \sin\theta\cos\theta = 0,$$

$$\hat{\varphi}\cdot\hat{\theta} = -\sin\varphi\cos\theta\cos\varphi + \cos\varphi\cos\theta\sin\varphi = 0$$

this coordinate system is orthogonal.

Tangents to curves

$\vec{c}(t) = (u(t), v(t), w(t))$ in u, v, w coordinates.

Then

$$\frac{d\vec{c}}{dt} = \frac{\partial \vec{c}}{\partial u}\frac{du}{dt} + \frac{\partial \vec{c}}{\partial v}\frac{dv}{dt} + \frac{\partial \vec{c}}{\partial w}\frac{dw}{dt}$$

$$= \vec{u}\frac{du}{dt} + \vec{v}\frac{dv}{dt} + \vec{w}\frac{dw}{dt}$$

$$= h_u \hat{u}\frac{du}{dt} + h_v \hat{v}\frac{dv}{dt} + h_w \hat{w}\frac{dw}{dt}$$

$$= h_u \frac{du}{dt}\hat{u} + h_v \frac{dv}{dt}\hat{v} + h_w \frac{dw}{dt}\hat{w}.$$

36.1 Example 3 Let

$\vec{c}(t) = 2\cos 3t\,\hat{\imath} + 2\sin 3t\,\hat{\jmath} + \hat{k}$. Find $\frac{d\vec{c}}{dt}$

in cylindrical coordinates.

Solution $\vec{c}(t) = (2\cos 3t, 2\sin 3t, 1)$ in x, y, z coordinates

$$\vec{c}(t) = \left(\sqrt{2^2\cos^2 3t + 2^2\sin^2 3t}, \arctan\left(\frac{2\sin 3t}{2\cos 3t}\right), 1\right)$$

$= (2, 3t, 1)$ in ρ, φ, z coordinates.

Then

$$\frac{d\vec{c}}{dt} = h_\rho \frac{d2}{dt}\hat{\rho} + h_\varphi \frac{d(3t)}{dt}\hat{\varphi} + h_z \frac{d1}{dt}\hat{z}$$

$$= 0 + \rho \cdot 3 \cdot \hat{\varphi} + 0 = 3\rho\hat{\varphi} = 6\hat{\varphi}.$$