Vector Calculus Lecture 2

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Continuity theorems

Theorem

(a) If f and g are continuous at $x=a$ then $f+g$ is continuous at $x=a$

(b) If f and g are continuous at $x=a$ then $f \cdot g$ is continuous at $x=a$

(c) If f is continuous at $x=a$ and $c \in \mathbb{R}$ then cf is continuous at $x=a$.

(d) If f and g are continuous at $x=a$ and $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at $x=a$.

(e) If f is continuous at $x=a$ and h is continuous at $x=f(a)$ then $h \circ f$ is continuous at $x=a$.

Theorem Polynomials, trig functions, exponentials, logs, $(\quad)^{1/n}$ and hyperbolic trig functions are all continuous <u>everywhere they are defined</u> <u>Caveat</u>

§1.2 Example 1 Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$

Is f continuous at $(x,y) = (0,0)$?

Solution: $f(x,y)$ is continuous at $(x,y) = (0,0)$ if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

$f(0,0)$ is not well defined, so
f is not continuous at $(0,0)$

Even worse, since

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist.

So f is really not continuous at $(x,y) = (0,0)$.

§1.2 Example 2: Let $f(x,y) = \log(1-xy)$.
Where is f continuous?

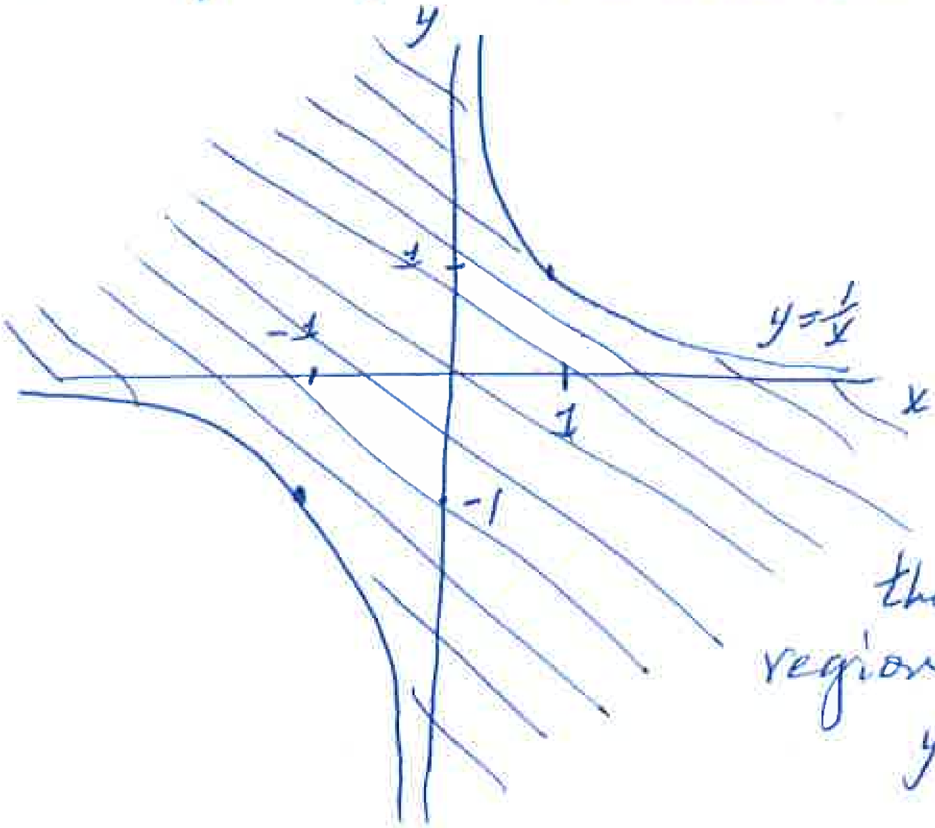
Solution: Since $1-xy$ is a polynomial, $1-xy$ is continuous everywhere.

$\log z$ is continuous everywhere it is defined.
$\log z$ is defined for $z \in \mathbb{R}_{>0}$.

So $f(x,y) = \log(1-xy)$ is continuous everywhere it is defined.

$\log(1-xy)$ is defined for $1-xy > 0$.

So $f(x,y) = \log(1-xy)$ is continuous for $xy < 1$



the shaded region is $y < \frac{1}{x}$.

§1.2 Example 3: Let

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}}, & \text{for } (x,y) \neq (0,0), \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Where is $f(x,y)$ continuous?

Solution: Since x^2, x^2+y^2 are polynomials and $(x^2+y^2)^{\frac{1}{2}} \neq 0$ for $(x,y) \neq (0,0)$ then

$$\frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \text{ is continuous for } (x,y) \neq (0,0)$$

(Note that $(x^2+y^2)^{\frac{1}{2}}$ is defined since $x^2+y^2 > 0$ when $(x,y) \neq (0,0)$)

Since $0 \leq \dfrac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \leq \dfrac{x^2+y^2}{(x^2+y^2)^{\frac{1}{2}}} = (x^2+y^2)^{\frac{1}{2}}$, then

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \leq \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$$

So $0 \leq \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \leq 0.$

So

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} = 0 = f(0,0).$$

So $f(x,y)$ is continuous for $(x,y) = (0,0)$.