Vector Calculus Lecture 23

§4.3 Example 6 Find the surface area of the cone parametrised by

$$\Phi(u,v) = (u\cos v, u\sin v, u)$$

for $0 \le v \le 2\pi$ and $0 \le u \le 1$.

Solution

$$\text{Surface area of } S = \iint_S dS = \iint_S |\vec{T}_u \times \vec{T}_v|\, du\, dv$$

Then

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 1)$$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u\sin v, u\cos v, 0)$$

and

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \begin{matrix} \hat{i}(0 - u\cos v) \\ -\hat{j}(0 + u\sin v) \\ +\hat{k}(u\cos^2 v + u\sin^2 v) \end{matrix}$$

and

$$|\vec{T}_u \times \vec{T}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$$

So

$$\iint_S |\vec{T}_u \times \vec{T}_v|\, du\, dv = \int_{v=0}^{2\pi}\int_{u=0}^{u=1} \sqrt{2}\, u\, du\, dv$$

$$= \int_{v=0}^{v=2\pi} \sqrt{2}\frac{u^2}{2}\Big]_{u=0}^{u=1} dv = \int_{v=0}^{v=2\pi} \frac{\sqrt{2}}{2}\, dv = \frac{\sqrt{2}}{2} 2\pi = \sqrt{2}\pi.$$

§4.3 Example 7 Find an expression for the surface area if $z = f(x,y)$.

Solution The surface is parametrized by

$$\Phi(x,y) = (x, y, f(x,y))$$

Then

$$\vec{T}_x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial f}{\partial x}\right) = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$\vec{T}_y = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial f}{\partial y}\right) = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

$$\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \begin{matrix} \hat{i}\left(0 - \frac{\partial f}{\partial x}\right) \\ -\hat{j}\left(\frac{\partial f}{\partial y} - 0\right) \\ +\hat{k}(1-0) \end{matrix} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$$|\vec{T}_x \times \vec{T}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

So

$$\text{Surface area} = \iint_S dS = \iint_S |\vec{T}_x \times \vec{T}_y| \, dx\, dy$$

$$= \iint_S \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dx\, dy.$$

§4.4 Example 1 Evaluate $\iint_S z^2 dS$, where S is the unit sphere centred at the origin.

Solution In spherical coordinates the sphere S has equation $r=1$.

So
$$x = r\cos\varphi\sin\theta = \cos\varphi\sin\theta$$
$$y = r\sin\varphi\sin\theta = \sin\varphi\sin\theta$$
$$z = \cos\theta$$
for the surface S and
$$\Phi(\varphi,\theta) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$$
for $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$

gives a parametrization of the sphere.

$$\vec{T}_\varphi = \left(\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}\right) = (-\sin\varphi\sin\theta, \cos\varphi\sin\theta, 0)$$
$$\vec{T}_\theta = \left(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}\right) = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)$$

$$\vec{T}_\varphi \times \vec{T}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\varphi\sin\theta & \cos\varphi\sin\theta & 0 \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \end{vmatrix}$$
$$= \hat{\imath}(-\cos\varphi\sin^2\theta - 0) - \hat{\jmath}(+\sin\varphi\sin^2\theta - 0)$$
$$+ \hat{k}(-\sin^2\varphi\sin\theta\cos\theta - \cos^2\varphi\sin\theta\cos\theta)$$

$$= (-\cos\varphi \sin^2\theta, -\sin\varphi \sin^2\theta, -\sin\theta\cos\theta)$$

$$|\vec{T}_\varphi \times \vec{T}_\theta| = \sqrt{\cos^2\varphi \sin^4\theta + \sin^2\varphi \sin^4\theta + \sin^2\theta\cos^2\theta}$$

$$= \sqrt{\sin^4\theta + \sin^2\theta\cos^2\theta} = |\sin\theta|\sqrt{\sin^2\theta + \cos^2\theta}$$

$$= |\sin\theta|$$

So

$$\iint_S z^2 dS = \iint_S z^2 |\vec{T}_\varphi \times \vec{T}_\theta| \, d\varphi \, d\theta$$

$$= \iint_S (\cos\theta)^2 \cdot |\sin\theta| \, d\varphi \, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \cos^2\theta \sin\theta \, d\varphi \, d\theta$$

$$= 2\pi \int_{\theta=0}^{\theta=\pi} \cos^2\theta \sin\theta \, d\theta$$

$$= 2\pi \left(-\frac{\cos^3\theta}{3}\right)\Big]_{\theta=0}^{\theta=\pi} = 2\pi\left(-\frac{\cos^3\pi}{3} - \left(-\frac{\cos^3 0}{3}\right)\right)$$

$$= 2\pi\left(-\frac{(-1)^3}{3} - \left(-\frac{1}{3}\right)\right) = 2\pi\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{4\pi}{3}.$$