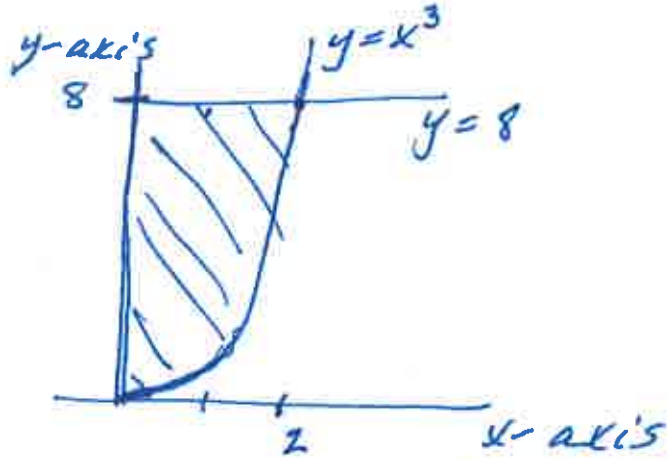
Vector Calculus Lecture 15

33.1 Example 5 Evaluate

$$\int_0^2 \int_{x^3}^8 \cos\left(y^{4/3}\right) dy\, dx$$

Solution:



$$\int_0^2 \int_{x^3}^8 \cos\left(y^{4/3}\right) dy\, dx = \int_{x=0}^{x=2} \int_{y=x^3}^{y=8} \cos\left(y^{4/3}\right) dy\, dx$$

$$= \int_{y=0}^{y=8} \int_{x=0}^{x=y^{1/3}} \cos\left(y^{4/3}\right) dx\, dy$$

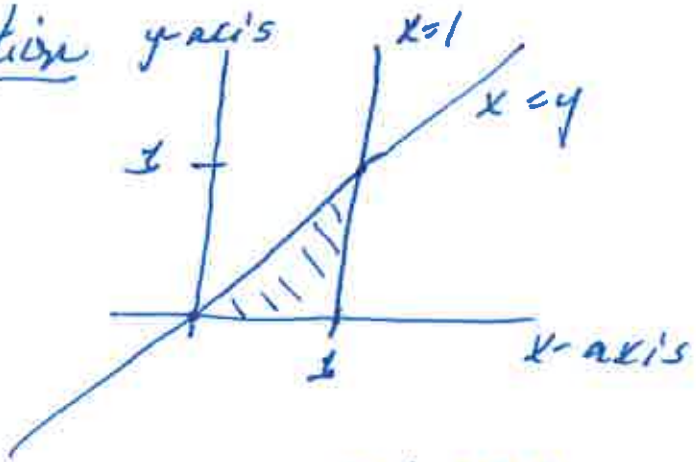
$$= \int_{y=0}^{y=8} \cos\left(y^{4/3}\right) x \Big|_{x=0}^{x=y^{1/3}} dy = \int_{y=0}^{y=8} \left(y^{1/3}\cos\left(y^{4/3}\right) - 0\right) dy$$

$$= \frac{3}{4} \sin\left(y^{4/3}\right) \Big|_{y=0}^{y=8} = \frac{3}{4}\sin\left(8^{4/3}\right) - \frac{3}{4}\sin 0$$

$$= \frac{3}{4}\sin\left(2^4\right) - 0 = \frac{3}{4}\sin 16$$

§3.1 Example 4 Evaluate $\int_0^1 \int_y^1 ye^{x^2} dx dy$.

Solution



$$\int_0^1 \int_y^1 e^{x^2} dx dy = \int_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} dx dy = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} dy dx$$

$$= \int_{x=0}^{x=1} e^{x^2} y \Big]_{y=0}^{y=x} dx = \int_{x=0}^{x=1} (xe^{x^2} - 0) dx$$

$$= \frac{1}{2} e^{x^2} \Big]_{x=0}^{x=1} = \frac{1}{2} e^1 - \frac{1}{2} e^0 = \frac{1}{2} e - \frac{1}{2}$$

$$= \frac{1}{2}(e-1).$$

§3.2 Example 1 Evaluate

$$\iiint_B (x+2y+3z)\,dV, \quad \text{where}$$

$B = [0,1] \times [-\frac{1}{2}, 0] \times [0, \frac{1}{3}]$.

Solution:

$$\iiint_B (x+2y+3z)\,dV = \int_{z=0}^{z=\frac{1}{3}} \int_{y=-\frac{1}{2}}^{y=0} \int_{x=0}^{x=1} (x+2y+3z)\,dx\,dy\,dz$$

$$= \int_{z=0}^{z=\frac{1}{3}} \int_{y=-\frac{1}{2}}^{y=0} \frac{x^2}{2} + 2yx + 3zx \Big]_{x=0}^{x=1} dy\,dz$$

$$= \int_{z=0}^{z=\frac{1}{3}} \int_{y=-\frac{1}{2}}^{y=0} \left(\left(\tfrac{1}{2}+2y+3z\right) - (0+0+0)\right) dy\,dz$$

$$= \int_{z=0}^{z=\frac{1}{3}} \tfrac{1}{2}y + y^2 + 3zy \Big]_{y=-\frac{1}{2}}^{y=0} dz$$

$$= \int_{z=0}^{z=\frac{1}{3}} \left((0+0+0) - \left(\tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + 3z\cdot\tfrac{1}{2}\right)\right) dz$$

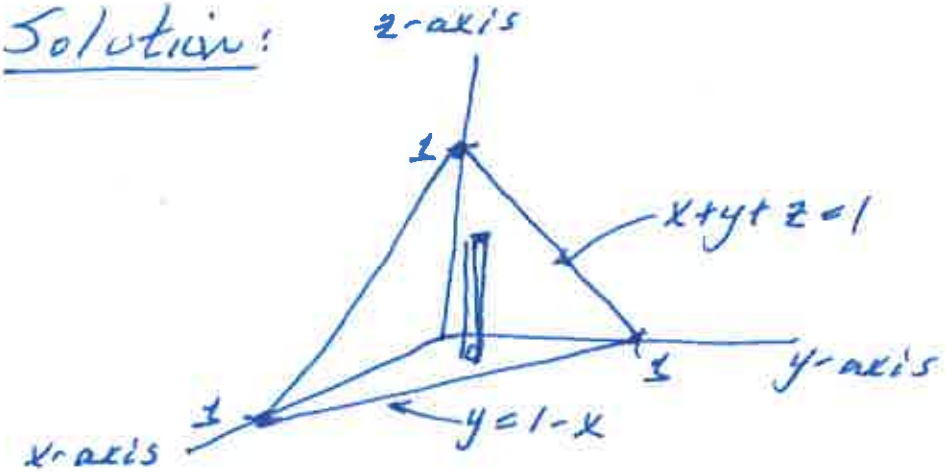
$$= \int_{z=0}^{z=\frac{1}{3}} \frac{3}{2} z\,dz = \frac{3}{2}\frac{z^2}{2} \Big]_{z=0}^{z=\frac{1}{3}} = \frac{3}{2}\frac{\frac{1}{3}\cdot\frac{1}{3}}{2} - 0$$

$$= \frac{\frac{1}{3}}{4} = \frac{1}{12}.$$

§3.2 Example 5 Evaluate

$$\iiint_D xy\,dV,$$ where D is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Solution:



$$\iiint_D xy\,dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} xy\;dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} zxy\Big]_{z=0}^{z=1-x-y}\;dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \left(xy(1-x-y)-0\right)\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \left(xy-x^2y-xy^2\right)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left. x\frac{y^2}{2} - x^2\frac{y^2}{2} - x\frac{y^3}{3} \right|_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{2}x(1-x)^2 - \frac{x^2}{2}(1-x)^2 - \frac{1}{3}x(1-x)^3 - (0-0-0) \right) dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{2}x(1-2x+x^2) - \frac{1}{2}x^2(1-2x+x^2) - \frac{1}{3}x(1-3x+3x^2-x^3) \right) dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3 - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^4 - \frac{1}{3}x + x^2 - x^3 + \frac{1}{3}x^4 \right) dx$$

$$= \int_{x=0}^{x=1} \left(\left(\frac{1}{2}-\frac{1}{3}\right)x + \left(-1-\frac{1}{2}+1\right)x^2 + \left(\frac{1}{2}+1-1\right)x^3 + \left(-\frac{1}{2}+\frac{1}{3}\right)x^4 \right) dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{6}x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 \right) dx$$

$$= \left. \frac{1}{6}\frac{x^2}{2} - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{x^4}{4} - \frac{1}{6}\frac{x^5}{5} \right|_{x=0}^{x=1}$$

$$= \left(\frac{1}{12} - \frac{1}{6} + \frac{1}{8} - \frac{1}{30}\right) - (0-0+0-0)$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{3} + \frac{1}{4} - \frac{1}{15}\right) = \frac{1}{2}\left(-\frac{1}{6} + \frac{11}{60}\right) = \frac{1}{2}\left(-\frac{10}{60} + \frac{11}{60}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{60} = \frac{1}{120}.$$