Vector Calculus Lecture 14

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①

§3.1 Example 1 Let $R = [-1,1] \times [0,1]$. Evaluate

$$\iint_R (x^2+y^2)\,dx\,dy$$

Solution

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{y=0}^{y=1} \int_{x=-1}^{x=1} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} \left(\frac{x^3}{3} + y^2 x \Big]_{x=-1}^{x=1} \right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{3}+y^2\right) - \left(\frac{(-1)^3}{3} + (-y^2)\right) \; dy$$

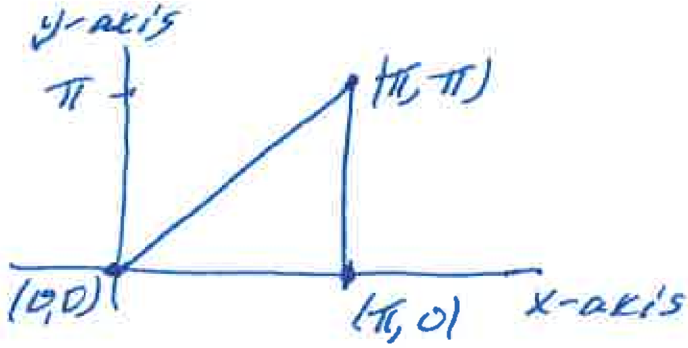
$$= \int_{y=0}^{y=1} \left(\frac{2}{3}+2y^2\right) dy = \frac{2}{3}y + \frac{2}{3}y^3 \Big]_{y=0}^{y=1}$$

$$= \left(\frac{2}{3}+\frac{2}{3}\right) - (0+0) = \frac{4}{3}.$$

§3.1 Example 2 Evaluate

$\iint_D (x^3y+\cos x)\,dA$, where D is the triangle with vertices at $(0,0)$, $(\pi,0)$ and (π,π).

Solution:



Using vertical strips

$$\iint_D (x^3y+\cos x)\,dA = \int_{x=0}^{x=\pi}\int_{y=0}^{y=x}(x^3y+\cos x)\,dy\,dx$$

$$=\int_{x=0}^{x=\pi}\left(\frac{x^3y^2}{2}+(\cos x)y\Big]_{y=0}^{y=x}\right)dx$$

$$=\int_{x=0}^{x=\pi}\left(\frac{x^5}{2}+x\cos x-(0+0)\right)dx$$

$$=\frac{1}{2}\frac{x^6}{6}+x\sin x+\cos x\Big]_{x=0}^{x=\pi}$$

since $\frac{d}{dx}(x\sin x)=x\cos x+\sin x$

so that $\frac{d}{dx}(x\sin x+\cos x)=x\cos x+\sin x+(-\sin x)$

$=x\cos x.$

So

$$\iint_D (x^3y + \cos x)\, dA = \frac{1}{12}x^6 + x\sin x + \cos x \Big]_{x=0}^{x=\pi}$$

$$= \left(\frac{1}{12}\pi^6 + \pi \sin\pi + \cos\pi\right) - \left(0 + 0 + \cos 0\right)$$

$$= \frac{1}{12}\pi^6 + 0 + (-1) - (0+0+1)$$

$$= \frac{1}{12}\pi^6 - 2.$$

<u>Using horizontal strips</u>

$$\iint_D (x^3y + \cos x)\, dA = \int_{y=0}^{y=\pi} \int_{x=y}^{x=\pi} (x^3y + \cos x)\, dx\, dy$$

$$= \int_{y=0}^{y=\pi} \left(y\frac{x^4}{4} + \sin x \Big]_{x=y}^{x=\pi} \right) dy$$

$$= \int_{y=0}^{y=\pi} \left(\left(y\frac{\pi^4}{4} + \sin\pi\right) - \left(\frac{y^5}{4} + \sin y\right) \right) dy$$

$$= \int_{y=0}^{y=\pi} \left(\frac{\pi^4}{4}y + 0 - \frac{y^5}{4} - \sin y \right) dy$$

$$= \frac{\pi^4}{4}\frac{y^2}{2} - \frac{y^6}{4\cdot 6} + \cos y \Big]_{y=0}^{y=\pi}$$

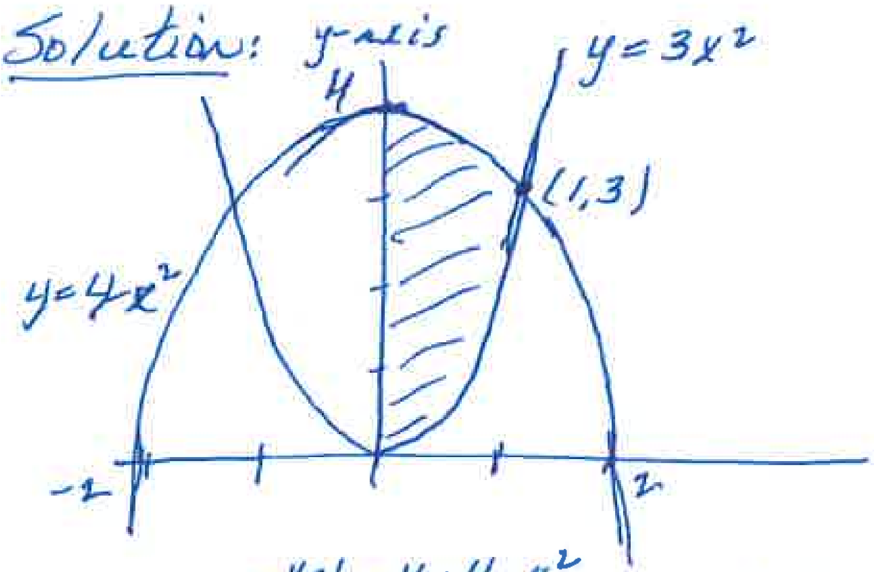
$$= \left(\frac{\pi^6}{8} - \frac{\pi^6}{24} + \cos\pi\right) - \left(0 - 0 + \cos 0\right)$$

$$= \frac{2}{24}\pi^6 + (-1) - (0-0+1)$$

$$= \frac{\pi^6}{12} - 2$$

<u>§3.1 Example 3</u> Find the area enclosed by $y = 3x^2$, $y = 4-x^2$ and the y-axis for $x \geq 0$.

<u>Solution:</u>



$$\text{Area} = \int_{x=0}^{x=1}\int_{y=3x^2}^{y=4-x^2} dy\,dx = \int_{x=0}^{x=1} y\Big]_{y=3x^2}^{y=4-x^2} dx$$

$$= \int_{x=0}^{x=1} \left((4-x^2) - (3x^2)\right) dx = \int_{x=0}^{x=1} (4-4x^2)\,dx$$

$$= 4x - 4\frac{x^3}{3}\Big]_{x=0}^{x=1} = \left(4 - 4\tfrac{1}{3}\right) - (0-0)$$

$$= 4 - \frac{4}{3} = \frac{8}{3}$$